A Survey of Logic 1

- why logic?
- rules
- presentations
- languages

'It is reasonable to hope that the relationship between computation and *mathematical* logic will be as fruitful in the next century as that between analysis and physics in the last. The development of this relationship demands a concern for both applications and mathematical elegance.'

John McCarthy

<u>A concern for</u>:

- applications
- mathematical elegance
- *philosophical acceptability*

Why not just classical logic?

(One argument)

We have two choices with respect to knowledge representation and manipulation:

Regimentation → (Quine) → Canonical language + logic

- *predicate logic*
- *Prolog*
- *constructive type-theory*

Autonomy → Tailored language + logic

A map

classical	modal	constructive
empire	*federation*	*"new"-empire*

(+ *"third world "* logics)

<u>modal (or intensional) realms</u>:

- *modal* *(necessity, possibility)*
- *tense* *(future, past)*
- *deontic* *(obligation, permission)*
- *epistemic* *(knowledge, belief)*
- *erotetic* *(questions, relevant reply)*

<u>third world realms</u>:

- non-monotonic
- *dialogue*

There are many ways in which we can attempt to lay out these "logics" in a common framework, *eg* :

- proof theory
- (abstract) model theory
- category theory

Proof theory is, perhaps, the most direct approach. It is based on the 🔑 idea of following a rule:

$$\frac{A \quad B}{A \wedge B} \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

So what are rules?

And what have they to do with reasoning?

What we are after is a notion of when some *assertion* ("A", say) follows from some *environment* ("Γ", say):

$$\langle \text{environment} \rangle \Vdash \langle \text{assertion} \rangle$$

We will use sentences to formalise assertions and consider one basic formalisation of the idea of an environment, namely a set of sentences. We write $\vdash$ instead of $\Vdash$

Call such a unit, $\Gamma \vdash A$, a <u>sequent</u>.

Let's consider three pictures as to what logic is about, and the form rules should take:

1) delimit/define language

2) sentences as the basic unit understood to carry meaning

3) test sentences against intuitions to establish which are valid, and which are invalid

4) characterise validity, " $\models A$ "

5) an <u>argument</u> is valid, if it doesn't proceed from valid assumptions to an invalid conclusion

6) intuitions may be in terms of an "algebraic model"

A semantic picture

(essentially switch 3 and 5)

3') test arguments : $\dfrac{\Gamma}{A}$

against intuitions, establish which are valid, which are invalid

4') characterise validity, " $\Gamma \models A$ "

5') a sentence, A, is valid when : $\models A$

Rules are on sentences (Prawitz)

3") test "rules" :

$$\frac{\Gamma_0 \vdash A_0 \quad \Gamma_1 \vdash A_1 \ldots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}$$

against intuitions, establishing those that are "valid", those that are "invalid"

4") characterise "validity"

1, 2 : as before

6 : may be forcibly rejected

Rules are on sequents (Gentzen)

Note: the shift from $\models$ *to* $\vdash$, *the bar is what is "semantic" in nature.*

Incorporation, into the picture, of:

1) computational aspects
   - *constructive notions*
   - *domain theory*
   - *insights from logic, functional and equational programming*
   - *real time considerations*

2) structural aspects
   - *notions of types*
   - *specification*

3) interactive aspects
   - *dialogue logic*
   - *adequate formal treatment of IO*
   - *man-machine interface*

A <u>rule</u>, then, is an expression of the form:

$$\frac{\Gamma_0 \vdash A_0 \quad \Gamma_1 \vdash A_1 \ldots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}$$

The sequents above the <u>bar</u> are called the <u>antecedents</u>, the sequent below the bar the <u>consequent</u> (of the rule).

A system of reasoning, a logic, is given by a language together with a collection of rules:

A <u>presentation</u>, P, is a finite collection of rules.

Note that a presentation, P, is just an "arbitrary" set of rules.

A <u>proof-schemata</u> in P is a tree such that:

1) every node is labelled with a sequent;

2) the sequent attached to a node must follow by *application* of one of the rules of P from the sequents attached to the nodes immediately above it;

(Leaves correspond to applications of rules without antecedents.)

If $\Gamma \vdash A$ is the root of such a tree, we say the tree proves the sequent $\Gamma \vdash A$.

Given a presentation P (over a language L) we write $\vdash_L{}^P$ for the relation generated by closing up under the notion of schematic proof. That is, $\Gamma \vdash_L{}^P A$ iff there is a schematic proof in P of $\Gamma \vdash A$.

Note, again, such a notion of proof "makes sense" for arbitrary collections of rules. Other foundational approaches to logic are not usually as broad.

There are many restrictions that we can impose on the form our rules may take. (Such restrictions or styles buy us different advantages and disadvantages.) For example:

- an axiomatic style, here the emphasis is on having axioms (rules with no antecedents) together with a small number of rules proper;

- a natural deduction style, here the rules come in pairs, an *introduction* and *elimination* rule for each operator;

- an all introduction rule style.

Notation:

$\Gamma, \Delta \vdash A$ for $\Gamma \cup \Delta \vdash A$

$\Gamma, B \vdash A$ for $\Gamma \cup \{B\} \vdash A$

Example:

Consider the following $\wedge, \rightarrow$ fragment:

Structural rules:

(Basic sequent) $A \vdash A$

(Thinning) $$\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}$$

Logical rules:

($\wedge$I) $$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}$$

($\wedge E_1$) $$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

($\wedge E_2$) $$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

($\rightarrow$I) $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

($\rightarrow$E) $$\frac{\Gamma \vdash A \quad \Delta \vdash A \rightarrow B}{\Gamma, \Delta \vdash B}$$

Example

$$A \to B, B \to C \vdash A \to C$$

Writing Γ for $\{ A \to B, B \to C \}$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \to B \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B} \qquad \Gamma \vdash B \to C}{\Gamma, A \vdash C}}{\Gamma \vdash A \to C}$$

<u>Note</u>:

- both top down (goal directed) and bottom up proof strategies "supported"
- the rule applied is usually obvious

Derived rules

A <u>derived rule</u>, R, of P is a rule s.t. that we have a proof schemata of the consequent of R from the presentation P augmented by the antecedents of R.

<u>Example</u>:

"I∧" is a derived rule of the above fragment:

$$\dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, B \vdash A \to C} \qquad \dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A}}{\dfrac{\dfrac{\Gamma, A \wedge B, B \vdash C}{\Gamma, A \wedge B \vdash B \to C} \qquad \dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash B}}{\Gamma, A \wedge B \vdash C}}$$

- Expressions (strings of symbols that have "meaning") are typed.

- Formation rules (ways of putting expressions together to give new expressions) can be considered as *functional* typing information.

- We call the types, syntactic categories.

- Syntactic categories may come furnished with both constants and (meta-)variables.

We will assume that a language, L, is comprised of at least two syntactic categories:

1) a set of *sentences* , sent(L);

2) a finite set of *operators*

$$op(L) = \{o_1, \ldots o_n\}$$

where each $o_i \in op(L)$ has:

degree : d_i

and *type* : $(\text{sentence})^{d_i} \rightarrow \text{sentence}$

Example:

The language of propositional logic has:

1) "no" other syntactic categories;

2) and op = $\{\wedge, \vee, \rightarrow, \leftrightarrow, \neg\}$

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For each of the following:

a) identify appropriate syntactic categories;

and for each such syntactic category:

b) suggest appropriate constants,

c) decide whether or not one is likely to need meta-variables.

1 - making assertions about imperative programs

2 - making assertions in an epistemic context

3 - making assertions in a deontic context

4 - Prolog

5 - making assertions about Prolog programs

6 - statements involving types in a functional programming language

A Survey of Logic 2

- consequence relations
- consistency
- modal systems

Should we regard any collection of rules as giving us a logic?

<u>Question</u>:

What makes $\vdash_L^P$ a logic?

<u>Answer</u>:

It has appropriate properties

A <u>property</u> of $\vdash_L^P$ is a "meta-rule", that is, a statement of the form:

if $\Gamma_0 \vdash_L^P A_0$ and ... $\Gamma_n \vdash_L^P A_n$ then $\Gamma \vdash_L^P A$

- clearly all rules give rise to meta-rules

- although the converse is not the case, we can always "safely" extend any presentation by adding any properties it may have.

Properties are established, in general, by inductive (meta)-proofs over the rules. Such proofs are really proof transformations.

rule of P $\Rightarrow$ derived rule of P $\Rightarrow$ property of P

atomic programs | *"simple" programs* | *recursive programs*

Reflexivity : $A, \Gamma \vdash A$

Monotonicity : $$\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}$$

Transitivity : $$\frac{\Gamma \vdash C \qquad C, \Delta \vdash A}{\Gamma, \Delta \vdash A}$$

Finiteness : $$\frac{\Gamma \vdash A}{\Gamma_0 \vdash A}$$

where Γ_0 is some finite subset of Γ

If $\vdash$ satisfies the above four properties we call it a consequence relation.

A complication

Strictly speaking, we should also impose a *substitutivity* property:

$$\frac{\Gamma \vdash A}{\Gamma^* \vdash A^*}$$

where $*: \text{sent}(L) \rightarrow \text{sent}(L)$

$*: o_i\,(A_1, \ldots A_{d_i}) \rightarrow o_i\,(A_1^*, \ldots A_{d_i}^*)$

and $\Gamma^* = \{ A^* \mid A \in \Gamma \}$

Presentations become "first-class" nations when we verify that they satisfy an *appropriate* collection of proprties, say:

reflexivity, monotonicity, transitivity

Example - "logics of negation"

$(\neg E) \quad \dfrac{\Gamma \vdash \neg A \qquad \Gamma \vdash A}{\Gamma \vdash B}$

$(\neg\neg E) \quad \dfrac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$

$(\neg I) \quad \dfrac{\Gamma, A \vdash \neg A}{\Gamma \vdash \neg A} \qquad \dfrac{\Gamma, \neg A \vdash A}{\Gamma \vdash A}$

$(RAA) \quad \dfrac{\Gamma, B \vdash \neg A \qquad \Gamma, B \vdash A}{\Gamma \vdash \neg B}$

"Close up" to get consequence relations then:

classical negation $= \neg E + \neg I \equiv \neg\neg E + RAA$

intuitionistic negation $= \neg E + RAA$

We say that a set of sentences, Γ, is:

simply consistent

iff $\neg \exists A$ s.t. $\Gamma \vdash A$ and $\Gamma \vdash \neg A$

absolutely consistent

iff $\exists A$ s.t. $\Gamma \nvdash A$

maximally simply consistent

iff 1) Γ is simply consistent

2) if $\Gamma \nvdash A$ then $\Gamma \cup \{A\}$ is not simply consistent

complete

iff $\forall A$ either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$

Result:

simply $\Rightarrow$ absolutely

(take any B then either $\Gamma \nvdash B$ or $\Gamma \nvdash \neg B$)

Result:

If $A, \neg A \vdash B$ then absolutely $\Rightarrow$ simply

(Suppose Γ not simply consistent,

so $\exists A$ s.t. $\Gamma \vdash A$ and $\Gamma \vdash \neg A$,

and by transitivity $\Gamma \vdash B$ (for arbitrary B)

so Γ not absolutely consistent)

Result:

not absolutely consistent $\Rightarrow$ complete

Another result

If Γ is maximally simply consistent

and $\vdash$ satisfies RAA

then Γ is complete

Proof:

Suppose Γ is not complete (we will argue by contradiction), so:

$\exists A$ s.t. $\Gamma \nvdash A$ and $\Gamma \nvdash \neg A$

Consider $\Gamma \cup \{A\}$ which is not simply consistent, so:

$\exists B$ s.t. $\Gamma, A \vdash B \quad \Gamma, A \vdash \neg B$

and hence by RAA $\Gamma \vdash \neg A$ (contradiction))

So what?

Exercise: Determine what principles of negation have been used at the meta-level in the above proofs.

Note:

- the interplay between
  1) definitions of consistency
  2) properties of negation ($\vdash$ - level)
  3) properties of negation at the meta-level

- since RAA $\Rightarrow$ $A, \neg A \vdash B$ *we might expect consistency to be a well-behaved notion in the presence of RAA*

We can extend our definitions of consistency of sets of sentences to consistency of (logic) presentations, P.

P is . . . <u>iff</u> the empty set of sentences, { } over $\vdash^{P}$ is . . .

(But note the definition for *complete* is inappropriate)

Research strategy - consider a notion (consistency, say) for classical logic and examine a number of equivalent variants. Establish what property "causes" the equivalence. Such properties "measure" how far we are from classical logic.

Hiz's presentation of "classical" logic:

$$\vdash \neg(A \to B) \to A \qquad \vdash \neg(A \to B) \to \neg B$$

$$\frac{\vdash A \to B \quad \vdash B \to C}{\vdash A \to C}$$

$$\frac{\vdash A \to (B \to C) \quad \vdash A \to B}{\vdash A \to C}$$

$$\frac{\vdash \neg A \to B \quad \vdash \neg A \to \neg B}{\vdash A}$$

<u>Result</u>: $\vdash^{\text{"usual"}} A \Leftrightarrow \vdash^{HIZ} A$

<u>But</u>: If we extend by adding $\vdash \neg A$, (HIZ* say), we <u>cannot</u> prove absolute inconsistency!

So . . .

$\vdash^{HIZ^*}$ is not simply consistent because we have $\vdash \neg A$ and $\vdash \neg(\neg A)$

hence $A, \neg A \nvdash B$ and hence RAA fails.

Justification: (of absolute consistency)

Consider the following matrices:

$\rightarrow$	t	T	f
t	t	f	f
T	t	t	t
f	t	t	t

	$\neg$
t	T
T	t
f	t

where "true" = {t, T}

Summary

Note the following:

$\vdash^{\text{"usual"}} A$	$\Leftrightarrow$	$\vdash^{HIZ} A$
$\Gamma \vdash^{\text{"usual"}} A$	?	$\Gamma \vdash^{HIZ} A$
rules$^{\text{"usual"}}$	$\not\Leftrightarrow$	rules^{HIZ}
properties$^{\text{"usual"}}$	$\not\Leftrightarrow$	properties^{HIZ}

We will assume that we are dealing with a language, L, suitable for sentential (propositional) logic, together with unary operators '$\Box$' and '$\Diamond$'.

First, we consider some properties on $\vdash$ that determine the "overall" behaviour of $\Box$.

Congruential : $$\frac{A \vdash B \quad B \vdash A}{\Box A \vdash \Box B}$$

Regular : $$\frac{\Gamma \vdash A \quad (\Gamma \neq \varnothing)}{\Box \Gamma \vdash \Box A}$$

Normal : $$\frac{\Gamma \vdash A}{\Box \Gamma \vdash \Box A}$$

Result: normal $\Rightarrow$ regular $\Rightarrow$ congruential

Result:

If . . . [Exercise: fill in the dots]

then normal iff

1) $$\frac{\vdash A}{\vdash \Box A}$$ (Necessitation)

2) $\vdash \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(Distribution)

Almost all work has been on normal systems.

It has become customary to give certain sentences names:

T	$\Box A \rightarrow A$
4	$\Box A \rightarrow \Box\Box A$
$\Diamond$defn.	$\Diamond A \leftrightarrow \neg \Box \neg A$
Euclidean	$\Diamond A \rightarrow \Box \Diamond A$
Brouwer	$A \rightarrow \Box \Diamond A$
McKinsey	$\Box \Diamond A \rightarrow \Diamond \Box A$
(Geach)	$\Diamond \Box A \rightarrow \Box \Diamond A$

We can use these sentences as axiom (schemata) to generate a range of modal systems.

K (for Kripke) = "usual" + Normal + $\Diamond$defn.

(K is the smallest normal system.)

"T" $= K + T$

$S_4 = K + T + 4$

$S_5 = K + T + 4 + B = K + T + 4 + E$

B $= K + T + B$

$S_{4.1} = S_4 + M$

$S_{4.2} = S_4 + G$

$S_{4.3} = S_4 +$

$(\Diamond A \wedge \Diamond B) \rightarrow (\Diamond(A \wedge B) \vee \Diamond(A \wedge \Diamond B) \vee \Diamond(\Diamond A \wedge B))$

$\vdash^{S4} \Diamond\Box\Diamond A \rightarrow \Diamond A$

1) $\Box\Diamond A \rightarrow \Diamond A$ [$\Diamond A$ for A in T]

...

2) $\Diamond\Box\Diamond A \rightarrow \Diamond\Diamond A$ [1, derived rule for T]

3) $\Box\neg A \rightarrow \Box\Box\neg A$ [$\neg A$ for A in 4]

...

4) $\neg\Diamond A \rightarrow \neg\Diamond\Diamond A$ [3, defn. of $\Diamond$]

5) $\Diamond\Diamond A \rightarrow \Diamond A$

6) $\Diamond\Box\Diamond A \rightarrow \Diamond A$ [2, 5]

It is customary to present modal systems via axioms rather than rules. Natural deduction based approaches don't always work very well.

What property then separates modal systems from classical logic?

<u>Answer</u>:

$\vdash$ is <u>smooth</u> <u>iff</u>

<u>if</u> $\dfrac{\vdash \Gamma}{\vdash A}$

<u>then</u> $\Gamma \vdash A$

<u>Examples</u>:

smooth: classical

"rough": modal, intuitionistic