Constructive Logic

- constructivity
- epistemic logic
- techniques

There are many ways to understand the difference between classical and constructive (intuitionistic, say) logic. We will consider two:

1) direct approach,

2) accommodate "constructivity" within the classical framework as the furnishing of extra information in a proof. (Nicolas Goodman)

A direct approach:

Intuitionistic logic "arose" as the logic of constructions:

- analyse the idea of a construction
- use it to "justify" rules

(of the logical constants):

We can introduce a "proof predicate", Π:

$\Pi(c, A)$: c is a constructive proof of A

where the constructions are understood as coming from a domain of constructions closed under:

a) application $c(c')$

b) pairing (and unpairing) $c = (c_0, c_1)$

We can now give clauses for interpreting the logical constants:

$\Pi(c, A \wedge B)$ <u>iff</u> $\Pi(c_0, A)$ and $\Pi(c_1, B)$

$\Pi(c, A \vee B)$ <u>iff</u> $\Pi(c_0, A)$ or $\Pi(c_1, B)$

$\Pi(c, \neg A)$ <u>iff</u> $\forall d.$

if $\Pi(d, A)$ then $\Pi(c(d), f)$

where f is some "contradiction", dependent on the domain *eg* over the natural numbers f might be $0=1$.

$\Pi(c, A \rightarrow B)$ <u>iff</u> $\forall d.$

if $\Pi(d, A)$ then $\Pi(c(d), B)$

<u>Note</u>:

$\wedge, \vee$ are "local"

$\neg$, are "global"

If we fix our domain of quantification, say to the natural numbers, then:

$\Pi(c, \exists x.\ A(x))$ <u>iff</u> $\Pi(c_0, A(\text{"}c_1\text{" for } x))$

$\Pi(c, \forall x.\ A(x))$ <u>iff</u> $\forall n.$

$\Pi(c(n), A(\text{"}n\text{" for } x))$

<u>Kreisel's modification</u>:

- add second constructions to $\rightarrow$, $\neg$, and $\forall$

ie $c = (c_0, c_1)$ where c_1 replaces c (on the right of the 'iff' above) and c_0 proves c_1 does the job.

Sundholm's analysis of this modification is that it guarantees the decidability of the Π predicate.

In this approach to "semantics" logical principles are justified by appeal to mathematical constructions, (in contrast to the usual justification of mathematics in terms of logic).

An all introduction rule presentation of intuitionistic logic is given by:

(∧I) $$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

(I∧) $$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

(∨I) $$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

(I∨) $$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

(→I) $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

(I→) $$\frac{\Gamma, B \vdash C \quad \Gamma, A \rightarrow B \vdash A}{\Gamma, A \rightarrow B \vdash C}$$

(¬I) $$\frac{\Gamma, A \vdash f}{\Gamma \vdash \neg A}$$

(I¬) $$\frac{\Gamma, \neg A \vdash A}{\Gamma, \neg A \vdash f}$$

(Basic sequent) $\Gamma, A \vdash A$

(Thinning) $$\frac{\Gamma \vdash f}{\Gamma \vdash A}$$

- the "extra" assumptions in (I→) and (I¬)

- no Δ's only Γ's

- this presentation (call it: Int) has the subformula property, giving us the basis for a decision procedure (for intuitionistic propositional logic).

- (Int is a consequence relation)

Suppose we consider a language without any "constant and function" symbols, *ie* any constants of syntactic category name,

<u>then</u> $\vdash \forall x.\ A(x)$ <u>iff</u> $\vdash \exists x.\ A(x)$

Suppose we consider a language where the only constant of syntactic category *name* is a unary one, g say,

<u>then</u> $\vdash \exists x.\ A(x)$ <u>iff</u> $\vdash \forall x.\ A(x)$
or $\vdash \forall x.\ A(g(x))$
or $\vdash \forall x.\ A(g(g(x)))$
or . . .

<u>Note</u>:The properties of the quantifiers depend closely on the linguistic resources available.

An alternative approach is to cope with constructivity within a modal extension of classical logic.

Philosophy:

- Constructivity is a matter of taste, not a matter of foundational correctness.

- Constructivity is a relative thing.

- See intuitionistic logic as a restriction of classical logic, introducing new logical constants to deal with constructive notions within the classical framework (here: epistemic S_4).

Result: There is a mapping (due to Godel) from Int to S_4 which is conservative in the sense that:

if Godel(Γ) $\vdash^{S4}$ Godel(A) then $\Gamma \vdash^{Int} A$

The mapping is given by:

$$
\begin{aligned}
\text{Godel}(A) &= \Box A \\
\text{Godel}(A \wedge B) &= \text{Godel}(A) \wedge \text{Godel}(B) \\
\text{Godel}(A \vee B) &= \text{Godel}(A) \vee \text{Godel}(B) \\
\text{Godel}(A \rightarrow B) &= \Box(\text{Godel}(A) \rightarrow \text{Godel}(B)) \\
\text{Godel}(f) &= f
\end{aligned}
$$

$$
\begin{aligned}
\text{Godel}(\exists x.\ A(x)) &= \exists x.\ \text{Godel}(A(x)) \\
\text{Godel}(\forall x.\ A(x)) &= \Box \forall x.\ \text{Godel}(A(x))
\end{aligned}
$$

Options:

epistemic operator	epistemic predicate
KA: A is ideally or potentially knowable or the stronger A is known	K(x): x can be observed or x can be constructed

There is a problem with taking 'KA' as 'A is known':

we don't necessarily want to claim that all the consequences of A are known if A is known. (Accommodating this takes us away from understood logical systems.)

Modal operators are, in general, examples of iterative concepts - it is meaningful for one operator to occur within the scope of another.

eg $K \neg K A$: "it is known that A is unknown"

- Predicates are inappropriate (as far as a surface level understanding is concerned) for iterative concepts because we are forced to "go" higher order.

- Modalities are inappropriate for non-iterative concepts (obligation ?) because they allow expressions to be well-formed that have no intuitive meaning.

"The" basic system for epistemic concepts can again be taken to be K, however an epistemic version of S_4 is often more appropriate:

KE: $KA \vdash A$

KI: $$\frac{K(\Gamma) \vdash A}{K(\Gamma) \vdash KA}$$

(Here presented in terms of rules rather than axioms; K rather than $\Box$)

Sentences that begin with a 'K' are called <u>epistemic</u>.

KE - "only true statements are knowable"

KI - expresses the deductive closure o knowability

<u>Results</u> (<u>Exercises</u>):

$\vdash KA \rightarrow A$

$\vdash KA \rightarrow KKA$

$\vdash KKA \rightarrow KA$

$$\frac{\vdash A}{\vdash KA}$$

$\vdash K(A \rightarrow B) \rightarrow (KA \rightarrow KB)$

(And: $\equiv S_4$)

There are two aspects of interpretation here, one in terms of our intuitions the other in terms of formal systems we already "understand".

We can attempt to justify the results of 15 directly:

eg $KA \rightarrow KKA$

"(If A is known then it is known that A is known);
if A is known then it is knowable that A is known;
so suppose that A is knowable, then A could become known, so it becomes knowable that A is known, hence the knowability of A is knowable."

A similar line of reasoning fails for:

$\neg KA \rightarrow K \neg KA$

because even if, A is not known we may come to know that it is not knowable (through self reflection), the unknowable sentences cannot be determined by self reflection.

known	*stronger than*	knowable
not known	*weaker than*	not knowable

So in adopting the "weaker" interpretation for K, we are adopting a "stronger" interpretation for $\neg$ K.

Disjunction

Notice the difference between:

$K(A \vee B)$: the disjunction is knowable

$KA \vee KB$: one of the disjuncts is knowable

$\vdash K(A \vee \neg A)$

$\nvdash KA \vee K\neg A$

The system we are considering has the disjunctive definabilty property:

$$\frac{\vdash KA \vee KB}{\vdash KA \text{ or } \vdash KB}$$

Existential definability

A similar situation arises for $\exists$:

$K\exists x.\, A(x)$: "I" know there is something with property A

$\exists x.\, KA(x)$: there is something "I" know has property A

We have the existential definabilty property:

$$\frac{\vdash \exists x.\, A(x)}{\vdash A(t)} \qquad \text{for some name t}$$

Techniques

How do we devise systems?

- **appeal to intuitions**
- **comparison with known systems**
- **identify 🔑 properties**

We will look at one technique to <u>enhance</u> this "methodology":

We can treat the named modal sentences as *rewrite-rules* , or *reduction laws* , statements as to when one string of modalities can be replaced by another.

A <u>modality</u> is a string of symbols over {□, ◇, ¬}.

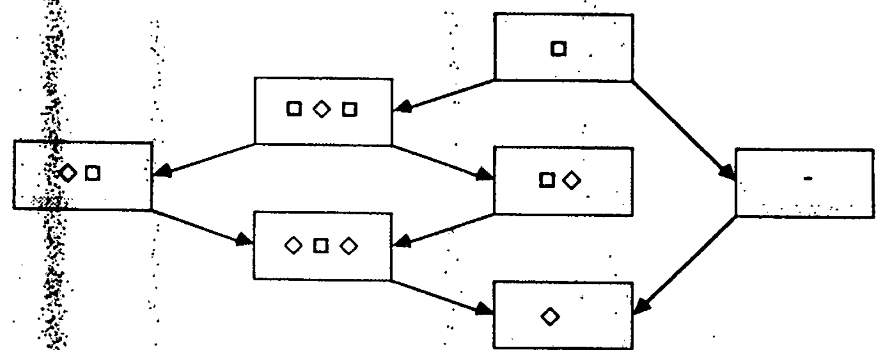
Modality diagrams

Given a system ⊢,

we write $s \to s'$ when $\vdash sA \to s'A$

We usually represent an equivalence class of modalities by its "simplest" member.

For example, in S_4 every modality is equivalent to one of:



or their negations.

Exercise: Investigate the claim that there are essentially only four kinds of negation:

$\neg A$	: A is "false"
$K \neg A$	: "I" know that A is "false"
$\neg KA$	: A is not knowable
$K \neg KA$	: the unknowability of A is knowable

and the relationship between them, by considering the modality diagram for S_4.

(Begin by replacing $\Box$ by K,
and $\Diamond$ by $\neg K \neg$.)

A sequent of the form $\vdash Qx_1, .. x_n. A(x_1, .. x_n)$, where $Qx_1, .. x_n$ is a string of quantifiers in the x_i's and A is quantifier free is called a prenex normal form.

There are intuitionistic sentences without equivalent prenex forms.

This is due to the failure of the converses of each of the following "reduction laws":

$\exists x. \neg A(x) \vdash \neg \forall x. A(x)$

$\forall x. A(x) \vee B \vdash \forall x. (A(x) \vee B)$

$\exists x. (A \rightarrow B(x)) \vdash A \rightarrow \exists x. B(x)$

$\exists x. (A(x) \rightarrow B) \vdash \forall x. A(x) \rightarrow B$

(use "modality" diagrams to see)

Dialogue Logic

- motivation
- logic of dialogue
  (the basic idea)
- a logic of commitment
  (one system in detail)

Motivation

Some of the originators + slogans:

(Wittgenstein):

meaning as use

employment of language is an activity

operational rather than denotational view

imperative rather than declarative view

Hintikka:

analysis of quantifiers of natural language

Lorenzen:

analysis of social action, ethics

foundations for intuitionistic logic

understanding the constructive notion of existence

. . .

Giles:

foundations of physics

experiments, testing

logic of commitment

Scott:

Giles' system as a logic of risk

cumulative consequence relation

Barth, Krabbe:

a Dutch school of dialogue logic

Relevance for computing science:

as an analysis of dialogue and dialogue management,

as an analysis of interaction

Logic of dialogue

Proof theory: a logic of dialogue

"Semantics": game theoretic

🔑 features:

- I play against you (nature)
- moves are assertions of sentences
  (at each position of the game some sentence has just been asserted)
- I win if I can produce a "true" atomic sentence
- You win if you can produce a "false" atomic sentence
- I start with some sentence
  (the one I am trying to "prove")
- who makes the next move is determined by the position

Rules

<u>Position</u>:	<u>Player</u>:	<u>Moves available</u>:
$A \vee B$	me	A, B
$A \wedge B$	you	A, B
$\exists x.\ A(x)$	me	A(n), n a name
$\forall x.\ A(x)$	you	A(n), n a name
$\neg A$	**	A

** Roles (me, you) are reversed, game continues

Write: $\models A$ just when I have a <u>winning strategy</u> starting with A.

A logic of commitment

Giles:

'meaning of any proposition is to be given in terms of an obligation that is incurred by him who asserts it'

Atomic formulas:

- correspond to the notion of a trial, experiment, or test,
- asserting an atomic formula, A, is an obligation to pay £1 should a trial of A yield the outcome "no".

If the outcome of such trials is always fixed, "what follows" reduces to classical logic

Rules (and evaluation functions)

Assert:	Obligation / "Value":
f	pay opponent £1 $\#(f) = 1$
$\neg A$	pay opponent £1, *if* he will assert A $\#(\neg A) = 1 - \#(A)$
$A \vee B$	assert A *or* assert B, own choice $\#(A \vee B) = \inf \{\#A, \#B\}$
A & B	assert A *or* assert B, opponents choice $\#(A \& B) = \sup \{\#A, \#B\}$

Why not the following rule for & ?

$$\text{assert } (A \,\&\, B) \equiv \text{assert } A, \ \text{assert } B$$

$$\#(A \,\&\, B) = \#(A) + \#(B)$$

Answer:

1) want: assert (A & A) ≡ assert A

2) analogous rule for ∀ wouldn't be viable

3) loss of limited liability

4) loss of symmetry with ∨

Assert: Obligation / "Value":

$\exists x. A(x)$ assert $A(n)$, for some n, own choice of n

$$\#(\exists x. A(x)) = \inf \{\#(A(n)) \mid n \in \text{names}\}$$

$\forall x. A(x)$ assert $A(n)$, for some n, opponents choice of n

$$\#(\forall x. A(x)) = \sup \{\#(A(n)) \mid n \in \text{names}\}$$

$A \to B$ assert B, if opponent asserts A

$$\#(A \to B) = \sup \{0, \#B - \#A\}$$

Arrow protocol

The obligation associated with asserting an arrow shouldn't bind one *forever* .

player 1:	player 2:
asserts $A \to B$	
	admits it
assertion	**annulled**
asserts $A \to B$	
	challenges it
	asserts A
asserts B	

Notes

1) $\to$ due to Lorenzen (dialogue interpretation of intuitionistic logic).

2) Asserting $\neg A$ incurs the same obligations as assereting $(A \to f)$.

3) $A \to B$ is <u>not</u> equivalent to $\neg A \vee B$, they incur different obligations:

for $A \to B$: you decide the outcome

for $\neg A \vee B$: I decide the outcome

$A, A \to B \vdash B$

$[\ \#(A) + \sup\{0, \#B - \#A\} \geq \#(B)\ ]$

$A \ \&\ (A \to B) \nvdash B$

$[\ \text{take: } \#(A) = 1/2,\ \#(B) = 1\ ]$

$$\frac{\Gamma, A \,\&\, B \vdash C}{\Gamma, A, B \vdash C} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \,\&\, B, A \,\&\, B \vdash C}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \,\&\, B \vdash C}$$ is not valid

$[\ \#(A) = \#(B) = 1/2,\ \#(C) = 3/4\ ]$

Recall the basic unit:

⟨environment⟩ $\vdash$ ⟨assertion⟩

and think of generalising:

database $\vdash$ query

to:

your assertions $\vdash$ my assertions

That is, rather than sets what we really want are bags, on both sides of the turnstile.

What kind of a "consequence" relation is $\vdash$?

Answer:

$\varnothing \vdash \varnothing$ "both risk nothing"

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta, A}$$ "both add A to what we risk"

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$ "you can risk more!"

$$\frac{\Gamma \vdash \Sigma \quad \Sigma \vdash \Delta}{\Gamma \vdash \Delta}$$ "cutting mutual risks Σ"

$$\frac{\Gamma, A \vdash \Delta, A}{\Gamma \vdash \Delta}$$ "both drop A from what we risk"

We can also add:

$$\frac{n.\Gamma \vdash n.\Delta}{\Gamma \vdash \Delta}$$ "scaling payoffs"

Result: (Scott)

For atomic formulas $A_1 \ldots A_n, B_1 \ldots B_m$

$A_1 \ldots A_n \vdash B_1 \ldots B_m$ follows from the above rules iff

$\forall\#$ s.t. $\#: \{A_1 \ldots B_m\} \to R^+$.

$\#(A_1) + \ldots + \#(A_n) \geq \#(B_1) \ldots \#(B_m)$

We can also give rules for the connectives:

(Exercise:)

Note that rules with two antecedents correspond to a choice.

Result: (an "almost" completeness result)

payoff from playing the game
almost(=)
payoff from #

(given #(A) for A atomic
= 1 - probability of success of trial of A

For any A, any $\varepsilon > 0$:

there is a strategy of debate s.t. from the starting position $\vdash A$ will guarantee a final position $\Gamma \vdash \Delta$ with:

$$\#(\Delta) - \#(\Gamma) < \#(A) + \varepsilon$$