SIGNATURE — SORTS, OPERATORS

1. Formal Def^n
2. Single Sorted

TERMS

Variables — EQUATIONS / FREE ALGEBRAS

Quotient Term Algebra
(Initial Algebra)

Definition (Signature)

A SIGNATURE
$$Sig = (S, Op)$$
comprises a set
+ S, the set of SORTS (symbols)

and a set
+ Op, the set of operator symbols.

The set Op is the union of pairwise disjoint subsets:

- K_s, set of constant symbols of sort $s \in S$

and

- $Op_{w,s}$ set of operator symbols with argument symbols $w \in S^+$ and range sort $s \in S$

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eg

$S = \{ Stack, Items \}$

$Op = \{ a, b, c, d, empty, push, pop, top \}$

$K_{stack} = ?$ $\quad K_{Items} = ?$

$K_{\langle Item, Stack \rangle, Stack} = \{ push \}$ $\quad$ etc

Single Sorted

S has only one element

eg 1 Counting numbers example

eg 2

Sorts Integer

Ops zero : Integer

plus : Integer, Integer → Integer

minus : Integer, Integer → Integer

negate : Integer → Integer

$K_{Integer} = \{zero\} = K_0$

$K_{\langle Integer, Integer \rangle Integer} = \{plus, minus\} = K_2$

$K_{\langle Integer \rangle Integer} = \{negate\} = K_1$

A single sorted signature $\Sigma = (s, \alpha)$ comprises

a sort name s (optional)

and a sequence

$\alpha = \langle K_0, K_1, K_2, \ldots, K_n, \ldots \rangle$

of pairwise disjoint sets of operator symbols

eg $h \in K_3$ is to be interpretted as an operator

$h : s, s, s \longrightarrow s$

$c \in K_0$ as

$c : \rightarrow s.$

or $c : s$

(2.5

Given a single sorted Signature

$$\Sigma = (s, \alpha)$$

$$\alpha = \langle K_0, K_1, K_2, \ldots K_n, \ldots \rangle$$

a MODEL $M = (s_M, \alpha_M)$

comprises a set S_M (the carrier set)

and a sequence

$$\alpha_M = \langle K_0^M, K_1^M, K_2^M, \ldots, K_n^M, \ldots \rangle$$

where $K_i^M = \{ N_M : N \in K_i \}$

and each N_M is a function

$$N_M : \overbrace{S_M \times S_M \times \ldots \times S_M}^{i \text{ copies}} \longrightarrow S_M$$

$$N_M : \prod_{j=1}^{i} S_M \longrightarrow S_M$$

Note

$$S_M \times S_M \times S_M = \{ (a, b, c) : a \in S_M, b \in S_M, c \in S_M \}$$

<u>example</u>

$\Sigma = (s, \alpha)$ where $\alpha = \langle K_0, K_1, \ldots \rangle$

$K_0 = \{ a, b, c, d \}$

$K_2 = \{ \_+\_ \}$

[for infix not" ie. a+b]

$K_1 = K_3 = K_4 = \ldots = \emptyset$

M_1

$S_M = \{ \ldots\ldots, -2, -1, 0, 1, 2, 3, \ldots \}$

$a_M = b_M = c_M = d_M = 1$

$+_M =$ "+" the usual addition

We can construct a model from any non-empty set with a diadic ("binary") operation

Why non-empty ??

Def^n TERMS

Given a signature

$$\Sigma = (S, \langle K_0, K_1, \dots K_n, \dots \rangle)$$

The following are terms

(i) every element of K_0 is a term

(ii) if $t_1, t_2, \dots, t_n$ are terms and f is an element of K_n then $f(t_1, t_2, \dots, t_n)$ is a term.

There are no other terms.

ex $K_0 = \{a, b, c, d\}$, $K_1 = \emptyset$, $K_2 = \{\_+\_\}$, $K_i = \emptyset$ $i > 2$

$a, b, c, d,$ are all terms

thus $(a+b)$, $(a+c)$ $(b+d)$, ... are all terms

bracket infix

thence $((a+b)+d)$ $(a+(b+c))$ etc.

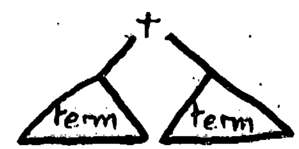
example ctd

$K_0 = \{a, b, c, d\}$ $K_2 = \_+\_$

Sorts	S
operators	$\_+\_ : S, S \to S$
No equations	

terms are

1. a, b, c, d

2. 

$K_0 = \{zero\}$ $K_1 = \{succ\}$, $K_i = \emptyset$ $i > 1$

terms are

Ex

Propositional logic I//

Sorts Proposition

Operators

$P, Q, R, \ldots\ldots, S$: Proposition

true, false : Proposition

$\_ \Rightarrow \_$: Proposition, Proposition $\rightarrow$ Proposition

$\_ \wedge \_$: Proposition, Proposition $\rightarrow$ Proposition

$\_ \vee \_$: Proposition, Proposition $\rightarrow$ Proposition

⋮

Terms :

....

Can you generalize this to the many sorted case?

eg

Sorts stack, Item

Operators

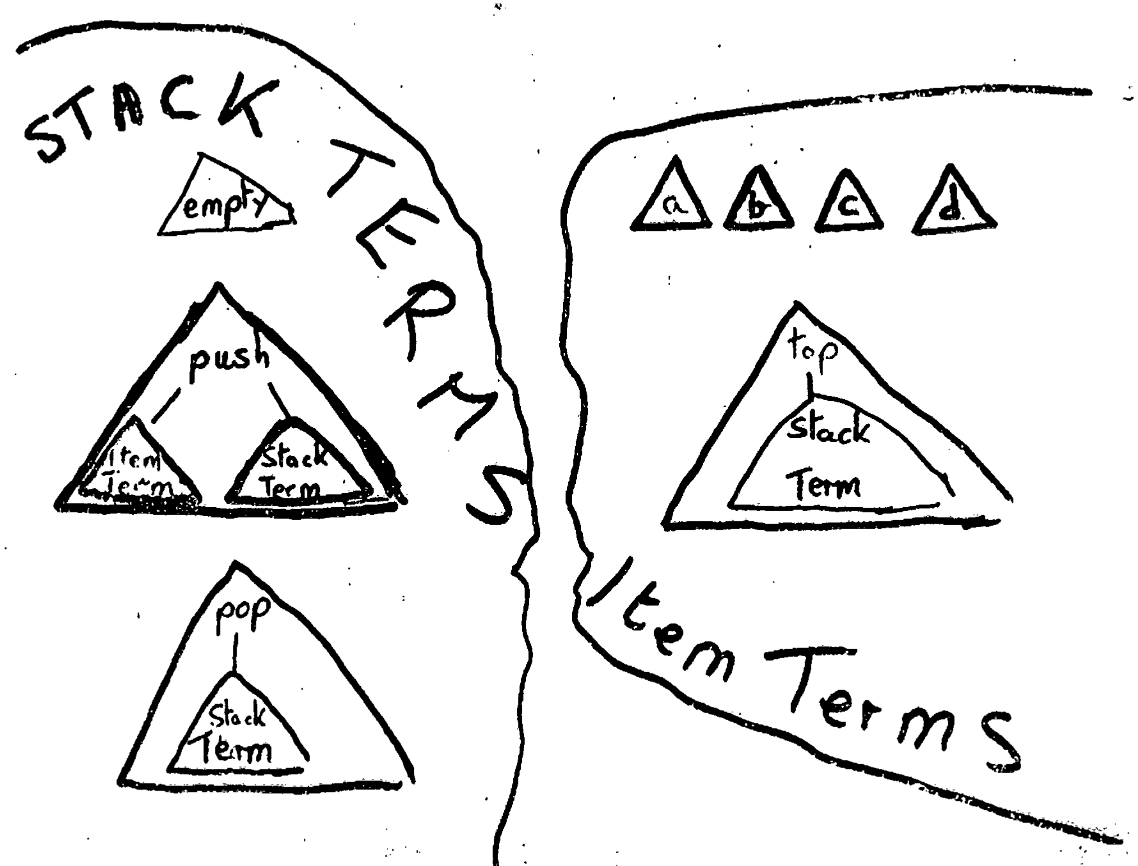
a, b, c, d : Item

empty : stack

push : Item, Stack $\rightarrow$ stack

pop : stack $\rightarrow$ stack

top : Stack $\rightarrow$ Item



Sorts ....

Operators

⋮

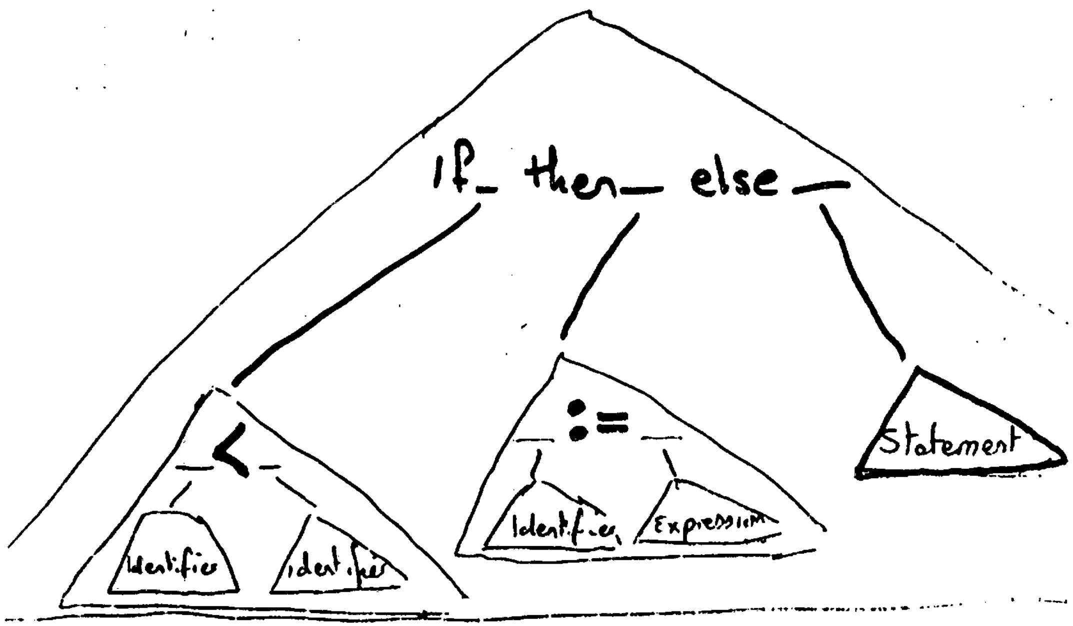
$\_ := \_$: Identifier, Expression $\rightarrow$ Statement

if _ then _ else _ : Expression, Statement, Statement $\rightarrow$ Statement

⋮

$<$: Identifier, Identifier $\rightarrow$ Expression

⋮



Given $\Sigma = (S, \langle K_0, K_1, \ldots, K_n \ldots \rangle)$

we have a set T_Σ of terms

and if $f \in K_n$ we have

$$f_T : T_\Sigma, T_\Sigma, \ldots, T_\Sigma \longrightarrow T_\Sigma$$

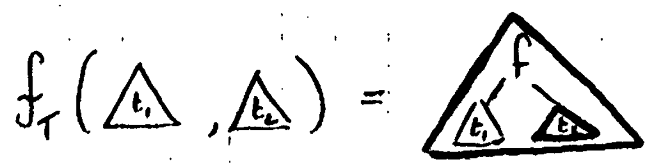
defined by

$$f_T(t_1, t_2, \ldots, t_n) = \underbrace{f(t_1, t_2, \ldots t_n)}_{\text{a term by definition part (ii)}}$$

n terms

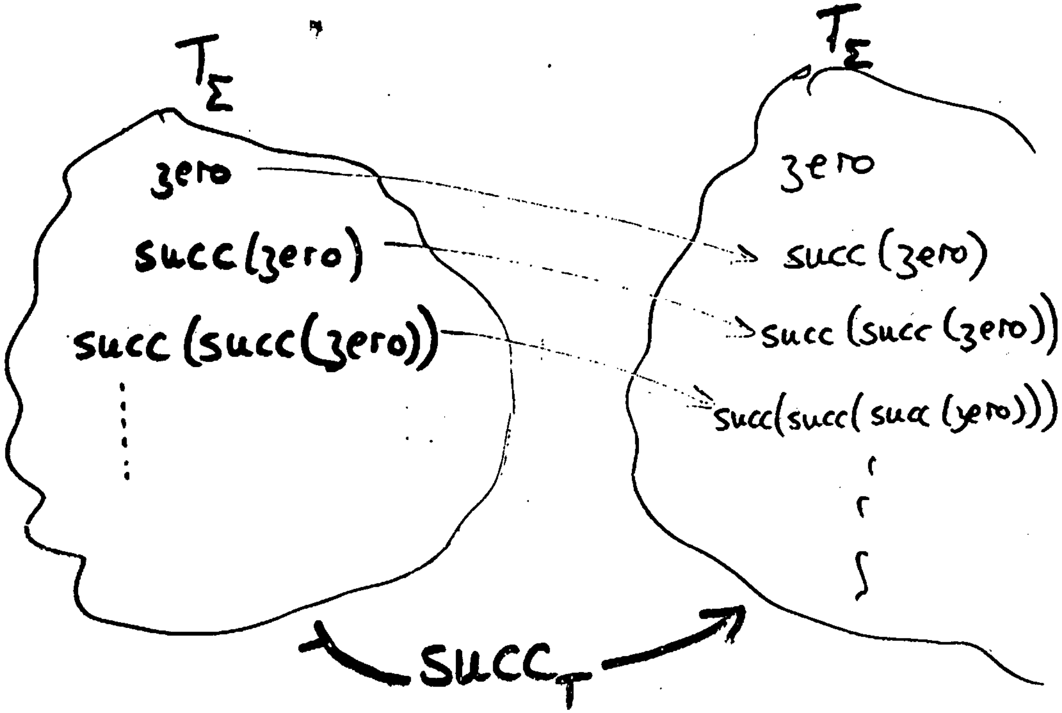
the function being defined

eg 1 // $f \in K_2$



Warning this is only intended to be illustrative!!!!

eg 2 $K_0 = \{zero\}$ $K_1 = \{succ\}$

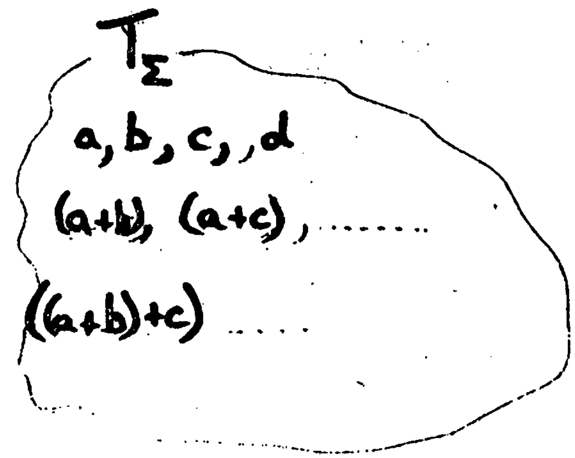


$succ_T : T_\Sigma \to T_\Sigma$

ex 5

$K_0 = \{a, b, c, d\}$

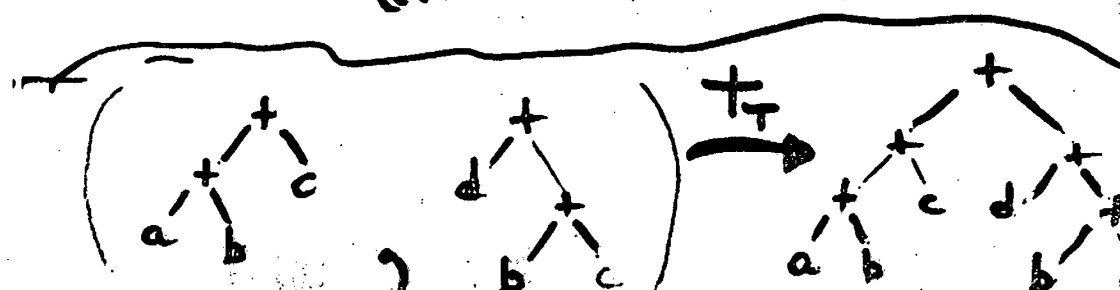
$K_2 = \{\_ + \_\}$



$$T_\Sigma \times T_\Sigma \xrightarrow{+_T} T_\Sigma$$

eg

$$((a+b)+c) +_T (d+(b+c))$$
$$= (((a+b)+c)+(d+(b+c)))$$



The Model described

with carrier set T_Σ

and operators f_T

is called the <u>Term Algebra</u>

over Σ and

will be denoted by T_Σ

(as well!)

Equations?

Let X be a set (of variables)

and $\Sigma = (S, \langle K_0, K_1, \ldots\ldots, K_n, \ldots \rangle)$

An equation over Σ (with variables in X)

is a pair (t_1, t_2)

of terms over

$(S, \langle K_0 \cup X, K_1, K_2, \ldots, K_n \rangle)$

<u>eg</u> $K_0 = \{a, b, c, d\}$ $K_2 = \{\_+\_\}$

$X = \{x, y, z, \ldots\ldots\}$

$(((x+y)+z), (x+(y+z)))$

is usually written

$((x+y)+z) = (x+(y+z))$

(associative law)

$((x+a), a)$

is usually written $x+a = a$

Notice

The set of terms over

$s, \langle K_0 \cup X, K_1, K_2, \ldots \rangle$

may also be considered as a Σ model.

It is called the **free** Σ model over X and denoted by

$T_\Sigma(X)$

(see previous example)

Satisfaction !

It is possible to define a notion of satisfaction whereby a **Model** may or may not satisfy an equation

This may be extended to sets of equations

Satisfaction?

eg $K_0 = \{a, b, c, d\}$
$K_2 = \{\_*\_\}$

$X = \{x, y, z\}$

equation

1 $$((x * y) * z) = (x * (y * z))$$

for a model to satisfy 1 we require the two sides to be equal if we uniformly substitute values from S_M for the variables. We require this for all such substitutions.

consider T_Σ then substitute

$a \to x$
$b \to y$
$c \to z$

are the two sides equal?

Quotient Term Algebra

$T_{\Sigma/E}$

sorts S

operators $a, b, c, d : S$

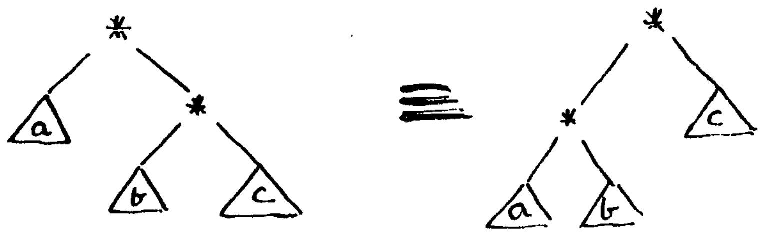
$\_ * \_ : S, S \to S$

Terms $\to$ **Trees** (non empty!)

Now add the equation

$x, y, z : S$

$$(x * y) * z = x * (y * z)$$



$a * (b * c)$ $\quad$ $(a * b) * c$

can remove bracketing

(non.empty) Lists / Sequences

sorts S

operators $a, b, c, d : S$

$\_ * \_ : S, S \to S$

equations $x, y, z : S$

$$(x * y) * z = x * (y * z)$$

$$(x * y) = (y * x)$$

$$\langle a, b, c, a \rangle = \langle a, a, b, c \rangle$$

etc

Bags (or Multi Sets)

sorts $\quad s$

operators $\quad a, b, c, d$

$\quad \_*\_ : S, S \to S$

equations

$x, y, z : S$

$$(x * y) * z = x * (y * z)$$
$$(x * y) = y * x$$
$$(x * x) = x$$

What is the quotient term algebra now?

sorts $\quad s$

operators $\quad a, b, c, d : S$

$\quad e : S$

$\quad * : S, S \to S$

equations

$x, y, z : S$

$$(x * y) * z = x * (y * z)$$
$$x * y = y * x$$
$$x * x = x$$

$$x * e = x$$
$$e * x = x$$

What is the quotient term algebra?

Ex Introduce e and the last two equations to the earlier examples.