Homomorphism

Simulation

MODEL → MODEL

(Slide Rule)

Evaluation

TERMS → MODEL

SYNTAX (LANGUAGE) → SEMANTICS (MEANING)

ASSIGNMENT

Free Algebra (Variables) → TERM ALGEBRA

Variable → term

$x \rightsquigarrow (a+b)+c$

Homomorphism
"SIMULATION"

VII	add	XIII	XX
|||||||	plus	∩|||	∩∩
7	+	13	20
↓ translate		↓ Translate	↓ translate
translate(7)		translate (13)	translate (20)

Computer Arithmetic

SIMULATION of "ONE MODEL"

by ANOTHER

7 representation_of (7) = 111

4 representation_of (4) = 100

$\mathbb{N}$, $+$ W, $+_w$

representation_of $(n+m)$

$=$ representation_of (n) $+_w$

representation_of (m)

Homomorphism

Sorts stack item

operator a b c d : item

empty : stack

top : stack $\rightarrow$ item

pop : stack $\rightarrow$ stack

push : item stack $\rightarrow$ stack

$$\text{stack}_1' \xrightarrow{\text{rep.of.stack}} \text{stack}_2'$$

$$\text{item}_1' \xrightarrow[\text{rep.of.item}]{} \text{item}_2'$$

We demand that the operations correspond

Sorts A
operators $i : A$
$\_ * \_ : AA \to A$
$\_^{-1} : A \to A$
equations
$a, b, c : A$
$a * (b * c) = (a * b) * c$
$i * a = a$
$a * i = a$
$a * a^{-1} = i$
$a^{-1} * a = i$

GROUP Homomorphism

$\mathbb{R}^+$ +ve reals		$\mathbb{R}$ all reals
1	$\xrightarrow{\log_{10}}$	0
mult^n		$+$
α^{-1}		$(-a)$

$\log_{10}(1) = 0$
$\log_{10}(\alpha * \beta) = \log(\alpha) + \log(\beta)$
$\log_{10}(\alpha^{-1}) = -\log(\alpha)$

Theorem

let $G, *, i, \_^{-1}$
& $H, \cdot, e, \_'$
be groups
and let $f : G \to H$
be such that

$\forall g_1, g_2 \in G . \; f(g_1 * g_2) = f(g_1) \cdot f(g_2)$

(ie f preserves the diadic op.)

Then

$f(i) = e$

and

$\forall g \in G . \; f(g^{-1}) = (f(g))'$

ie f preserves identity and inverses

(Bona Board)

Machines / Homomorphism / Simulation

Sorts inalph, outalph, state

operators

next : inalph state $\rightarrow$ state

print : state $\rightarrow$ outalph

$$\mathrm{inalph}'_1 \xrightarrow{\theta_{inalph}} \mathrm{inalph}'_2$$

$$\mathrm{outalph}'_1 \xrightarrow{\theta_{outalph}} \mathrm{outalph}'_2$$

$$\mathrm{state}'_1 \xrightarrow{\theta_{state}} \mathrm{state}'_2$$

must preserve the operators.

$$\begin{tikzcd}
\mathrm{inalph}'_1 \times \mathrm{state}'_1 \arrow[r, "\mathrm{next}'_1"] \arrow[d, "\theta_{inalph} \times \theta_{state}"'] & \mathrm{state}'_1 \arrow[d, "\theta_{\cancel{inalph}\ state}"] \\
\mathrm{inalph}'_2 \times \mathrm{state}'_2 \arrow[r, "\mathrm{next}'_2"'] & \mathrm{state}'_2
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{state}'_1 \arrow[r, "\mathrm{print}'_1"] \arrow[d, "\theta_{state}"'] & \mathrm{outalph}'_1 \arrow[d, "\theta_{outalph}"] \\
\mathrm{state}'_2 \arrow[r, "\mathrm{print}'_2"'] & \mathrm{outalph}'_2
\end{tikzcd}$$

THEORY

$$\text{Algebra}_1 \xrightarrow{\text{HOMOMORPHISM}} \text{Algebra}_2$$

carrier sets $\longrightarrow$ carrier sets
↑
functions

Preserving operations

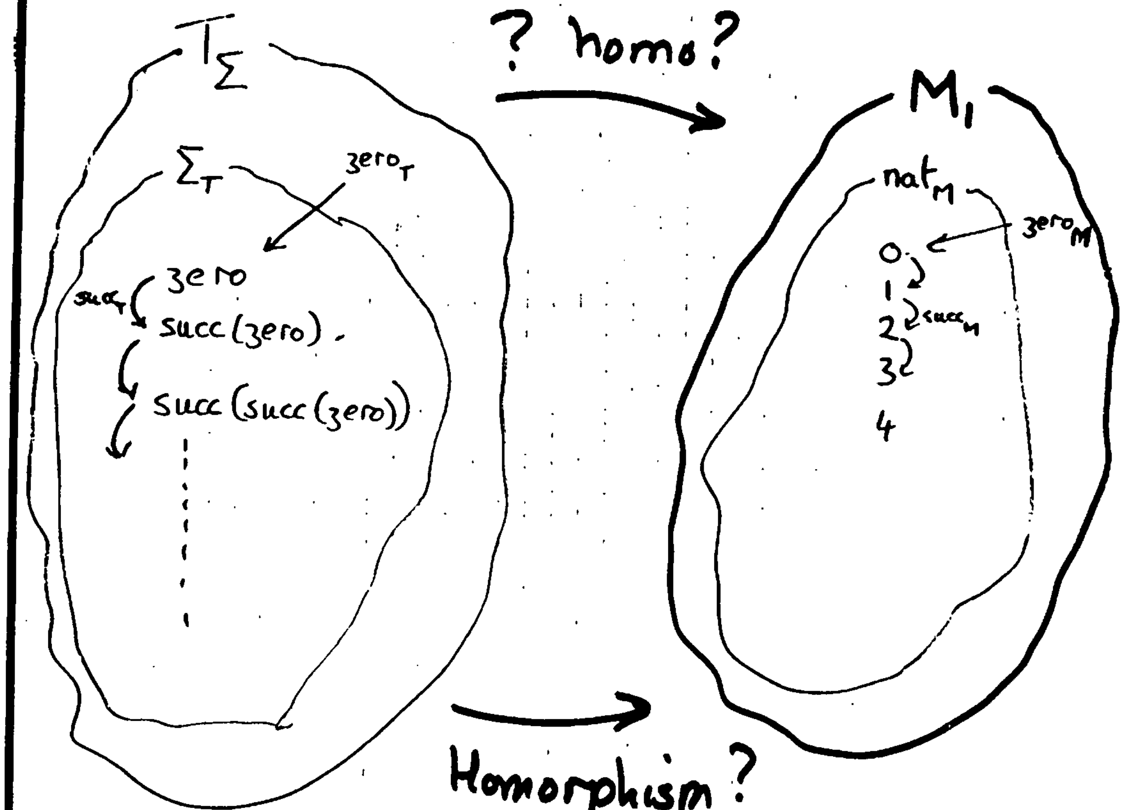
hence

Simulation

Experiment

$$\Sigma = (nat, \langle K_0, K_1, \emptyset, \emptyset \dots \rangle)$$

$$K_0 = \{zero\} \quad K_1 = \{succ\}$$



Let $\emptyset$ be a homorphism

$$\emptyset : T_\Sigma \longrightarrow M_1$$

What can it be ???

Definition (Evaluation of Terms)

Σ_T be the set of terms of a signature

$$\Sigma = (s, \langle K_0, K_1, \dots \rangle)$$

Let M be a model for Σ.

$$\mathrm{eval} : \Sigma_T \longrightarrow S_M \quad \text{is defined}$$

luation — terms — evaluate the term in the model. — carrier set of the Model

inductively. as follows

$$\underline{\mathrm{eval}(n)} = \underline{n_M} \qquad \text{for all } n \in K_0$$

$$\underline{\mathrm{eval}(f(t_1, \dots, t_r))} = \underline{f_M(\mathrm{eval}(t_1), \dots\dots, \mathrm{eval}(t_r))}$$

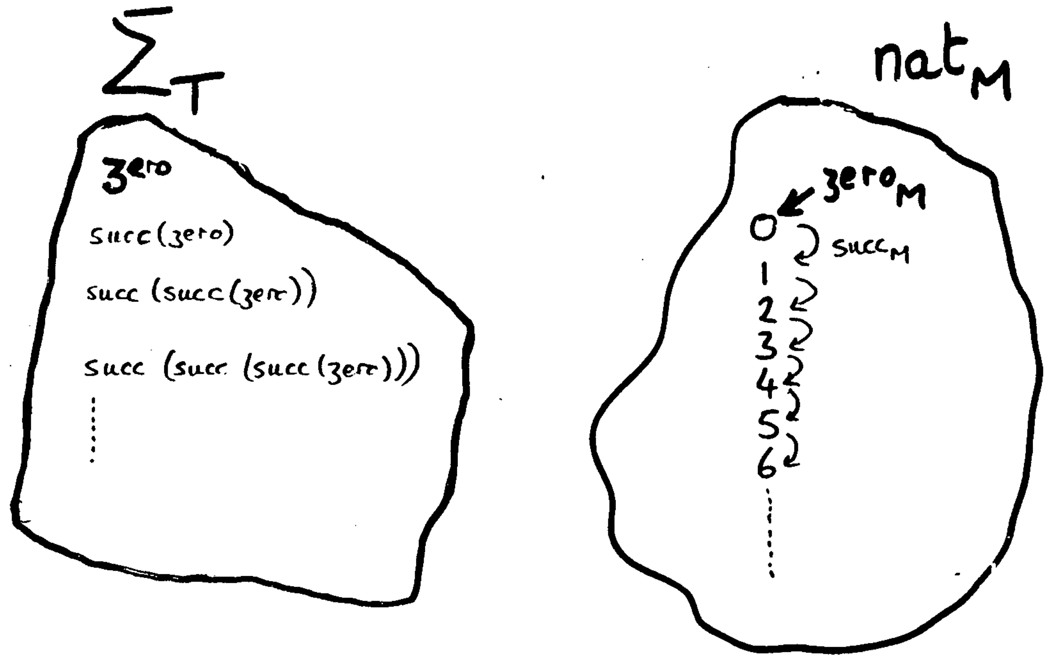
for all terms $t_1, \dots, t_r$

and $f \in K_r$ $(r > 0)$

Example

$$\Sigma = \{ nat, \langle \{zero\}, \{succ\}, \emptyset, \emptyset \dots\dots \rangle$$

sorts	nat
operators	zero : nat
	succ : nat $\to$ nat



$$\mathrm{eval}(zero) = zero_M = 0$$

$$\mathrm{eval}(succ(succ(zero)))$$

$$= succ_M(\mathrm{eval}(succ(zero)))$$

$$= succ_M(succ_M(\mathrm{eval}(zero)))$$

$$= succ_M(succ_M(0)) = succ_M(1) = 2$$

Fact:

eval is a homomorphism

Fact:

If M is a Σ model then eval is the only homomorphism from T_Σ to M

FACT! ie FACT!

If M is a Σ model there is a unique homomorphism from T_Σ to M

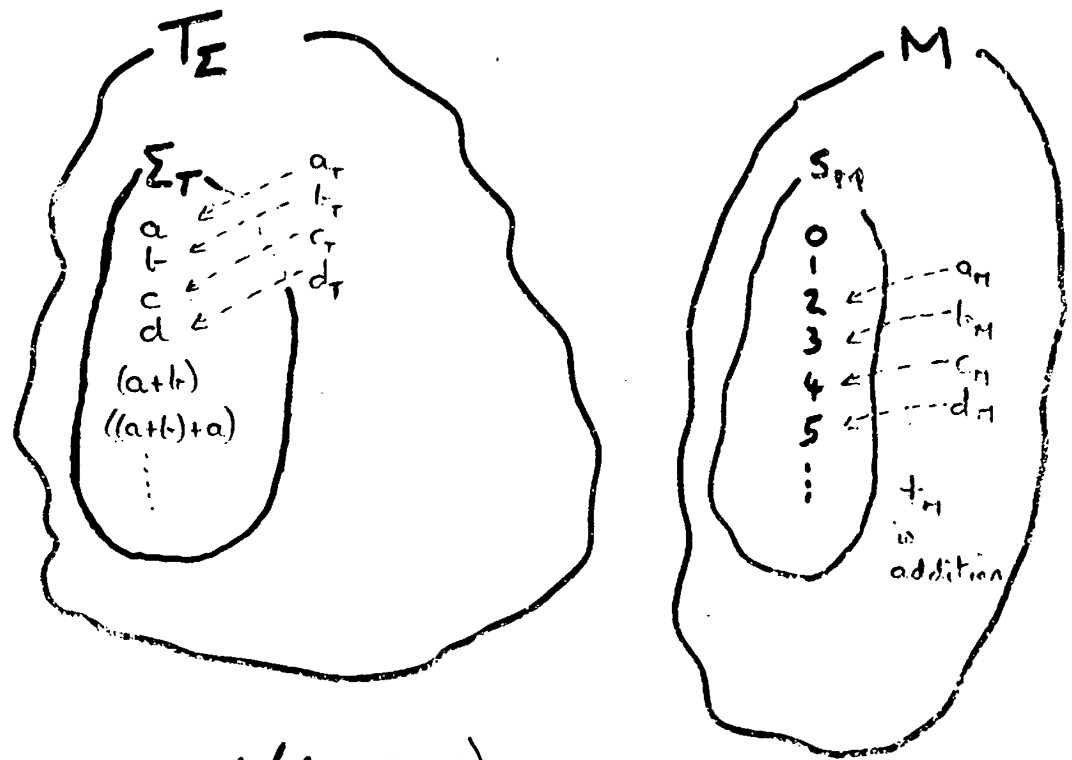
(called **eval**(uation))

Examples (eval) (3.1

$K_0 = \{a, b, c, d\}$

$K_2 = \{ \_+\_ \}$ $\quad K_1 = K_3 = \ldots = K_n = \ldots = \emptyset$

sorts S
operators a,b,c,d : S
_+_ : S,S →

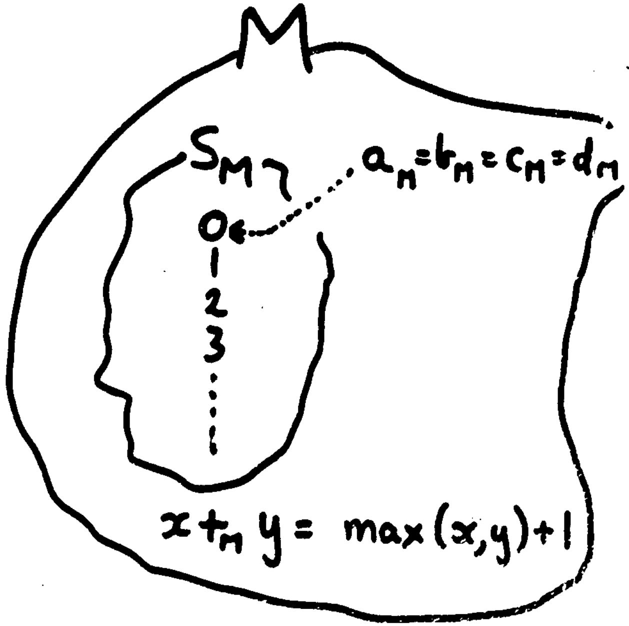


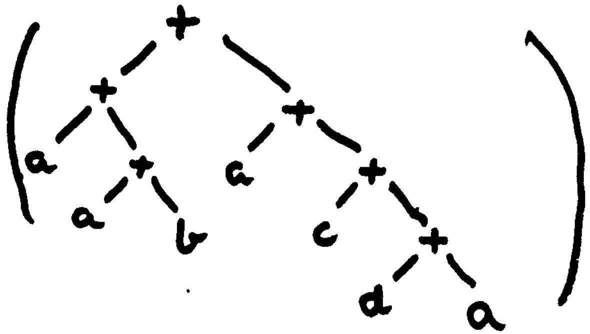
$$\mathrm{eval}(((a+b)+c))$$
$$= (\mathrm{eval}(a+b)) +_M (\mathrm{eval}\ c)$$
$$\vdots$$
$$= ((a_M +_M b_M) + c_M)$$
$$= ((2+3)+4) = 9$$

OK!

Eval

T_Σ as before

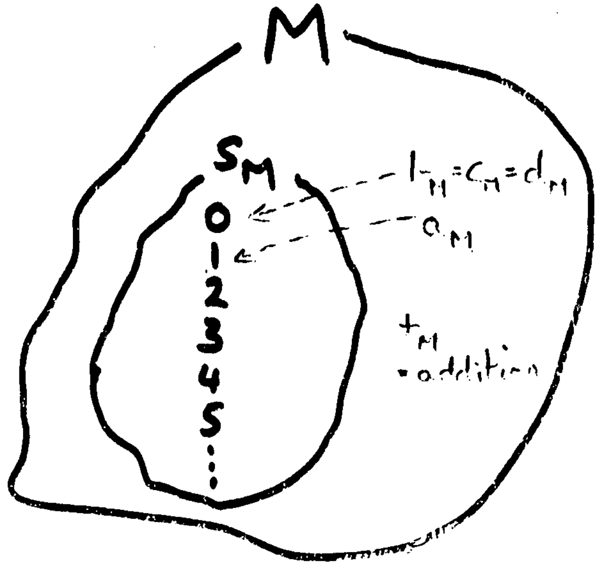


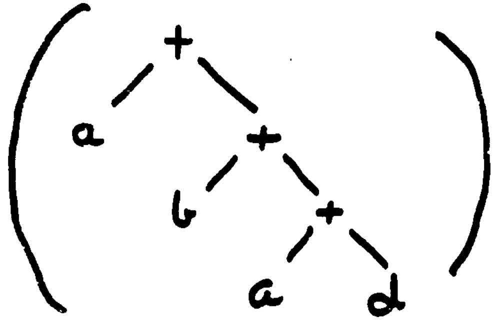
eval  = ?

$$\text{eval}\left(\left((a+(a+b))+(a+(c+(d+a)))\right)\right) = ?$$

Examples eval

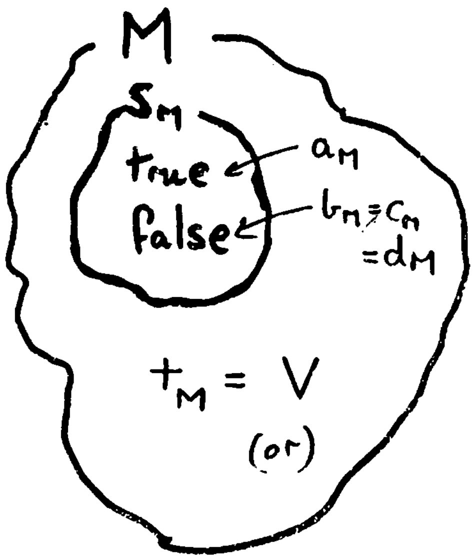
T_Σ as before



eval  = ?

Examples (eval)

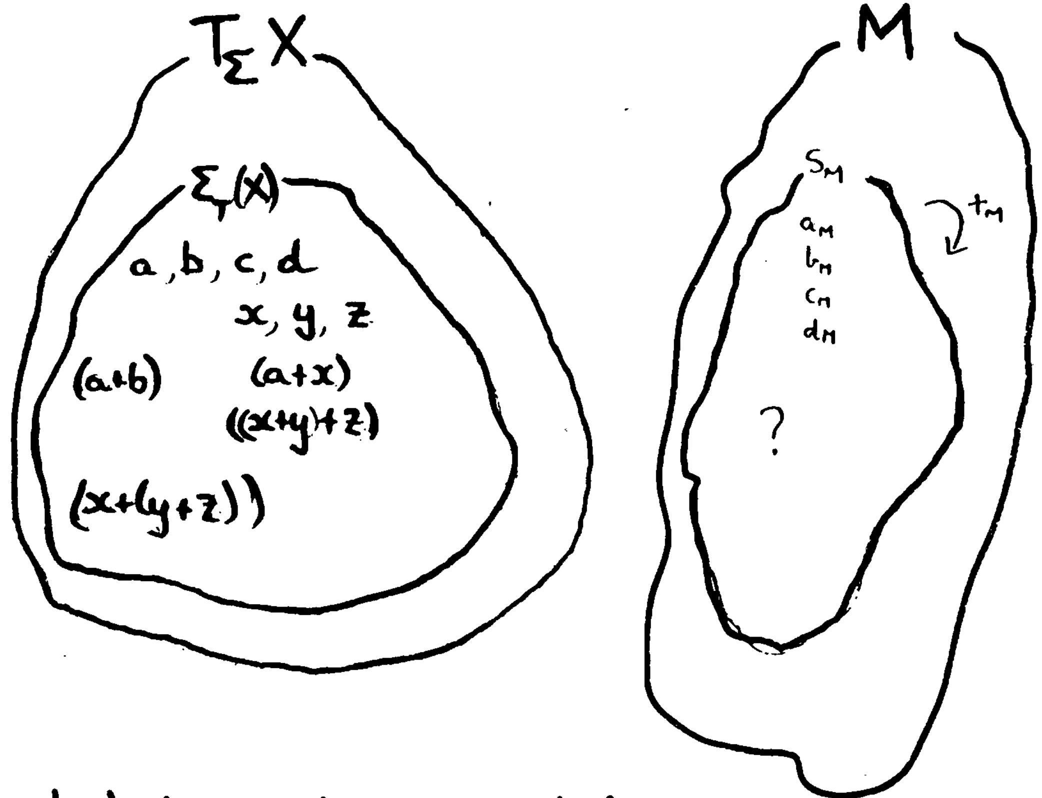
T_Σ as before



eval = ?

let $X = \{x, y, z\}$

$K_0 = \{a, b, c, d\}$ $\quad K_2 = \{\_+\_\}$



What is a homomorphism from $T_\Sigma X$ to a Σ model M?

Def^n A homomorphism ϕ from $T_\Sigma(X)$ to a Σ model A is called an assignment

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Let (α, β) be an equation over Σ with variables in X

Def A satisfies (α, β) precisely if $\phi(\alpha) = \phi(\beta)$ **for all** assignments $\phi : T_\Sigma(X) \to A$

ie "whatever assignments are made to the variables α & β will evaluate to equal values.

Ex

Sorts S

Operators $\_*\_ : S,S \to S$

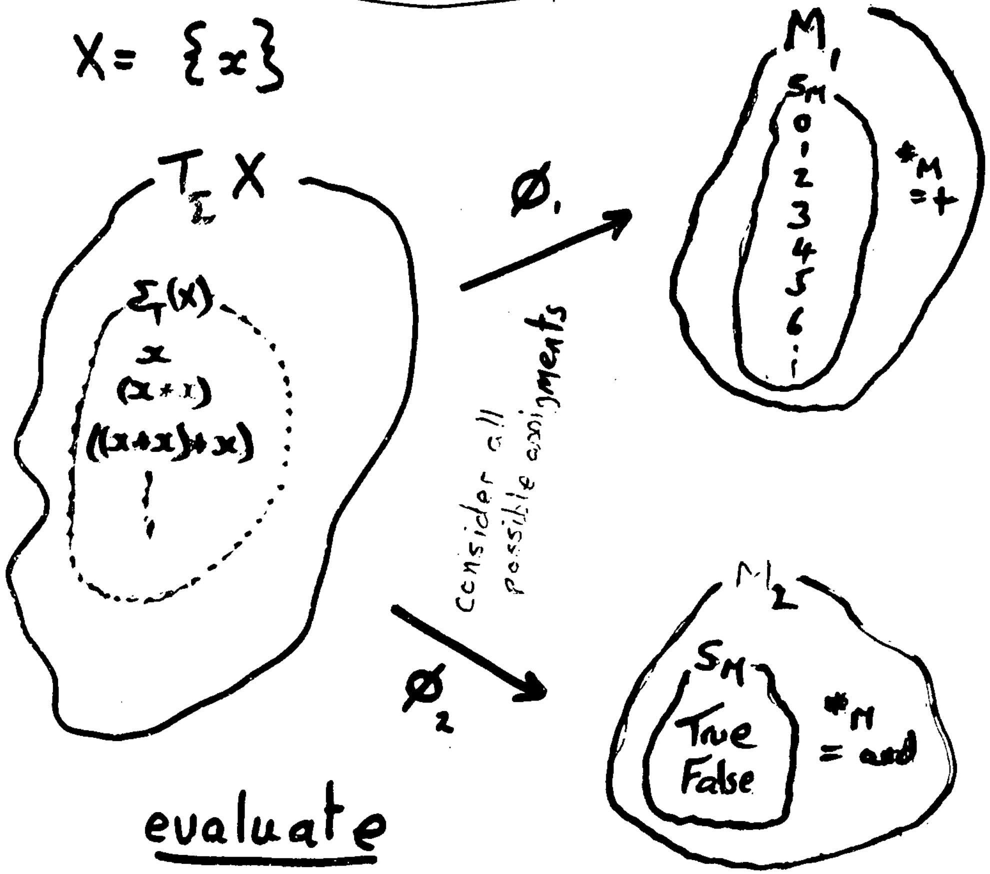
Equations $x : S$

$x * x = x$

Equation

$((x*x), x)$

$X = \{x\}$



evaluate

$\phi((x*x))$

and $\phi(x)$

Quotient Term Algebra

(3)11

Given a Theory Presentation

$$(\Sigma, E)$$

Let $t_1 \sim t_2$, $t_1, t_2 \in \Sigma_T$

if there is an assignment

$$\emptyset : T_\Sigma(X) \longrightarrow T_\Sigma$$

s.t. $\emptyset(\alpha) = t_1$

$\emptyset(\beta) = t_2$

and $(\alpha, \beta) \in E$

eg

Sorts S

operators $a, b, c, e : S$

$\_ * \_ : S, S \to S$

Equations $x : S$

$x * e = x$

$$c * e \xrightarrow{x \mapsto c} c$$

$$((a*b)*e)*e \xrightarrow{x \mapsto (a*b)*e} (a*b)*e \xrightarrow{x \mapsto (a*b)} (a*b)$$

$$((a*e)*b)*e \xrightarrow{x \mapsto (a*e)*b} (a*e)*b \xrightarrow{x \mapsto ?} (a*b)$$

? ?

$\_/\_ : num, num \to rational$

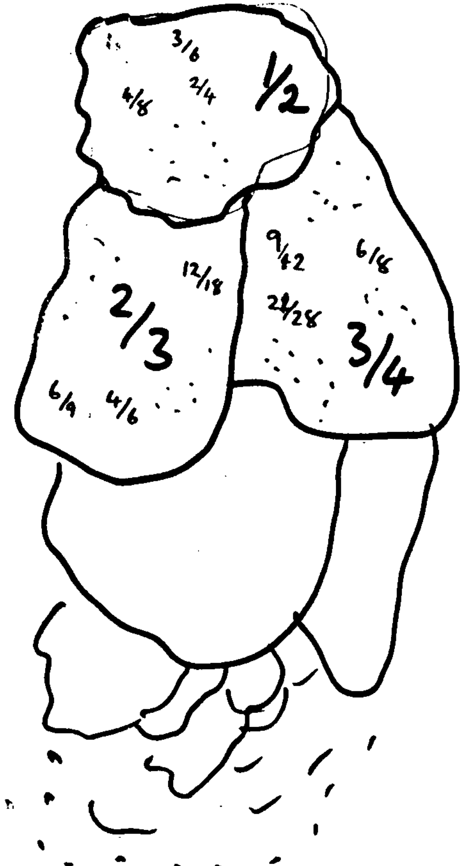
$\_+\_ : rational, rational \to rational$

equations?

$p/q = r/s$ if $ps = rq$

(i) $p/q + r/s = (ps + rq)/(qs)$

(ii) $p/q + r/s = (p+r)/(q+s)$

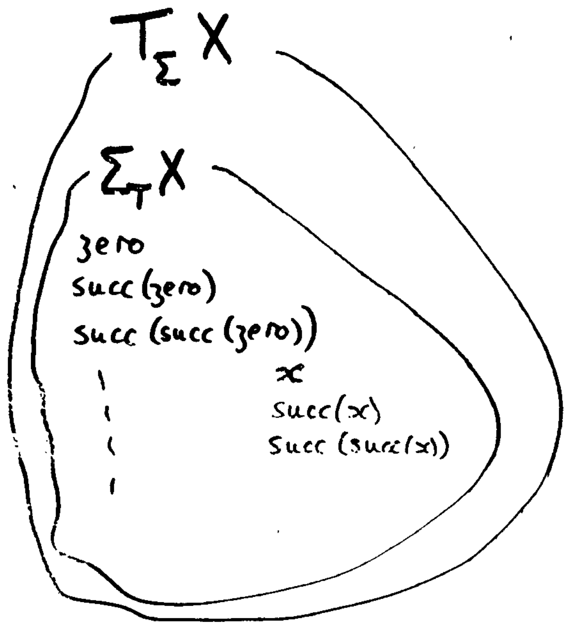


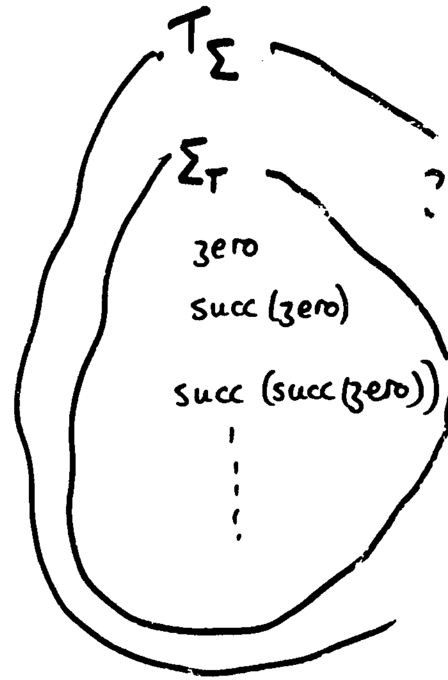
Example

$K_0 = \{zero\}$ $K_1 = \{succ\}$

$$\boxed{succ(succ(x)) = x}$$

$X = \{x\}$



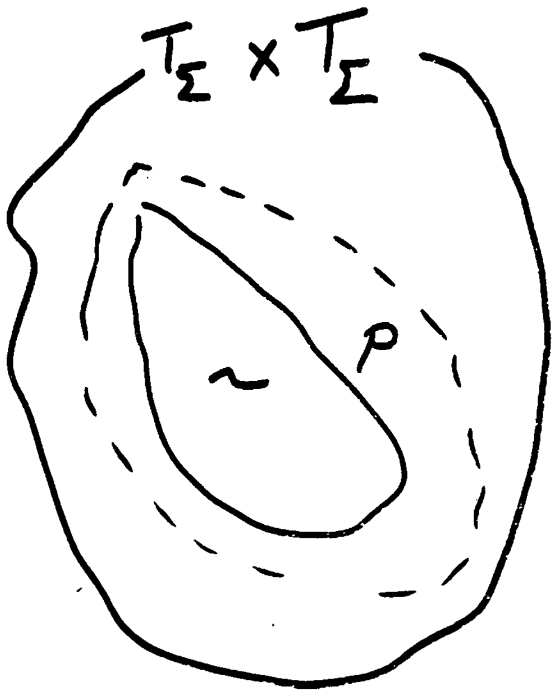


$\phi_1(x) = zero \Longrightarrow succ(succ(zero)) \sim zero$

$\phi_2(x) = succ(succ(zero))$

$\longrightarrow succ(succ(succ(succ(zero)))) \sim succ(succ(zer$

Let ρ be the smallest congruence on T_Σ containing $\sim$



Def^n Let Σ_T/E denote the set of equivalence classes, Σ_T/ρ

For each $f \in K_n$

we have $f_T : \Sigma_T \times \ldots \times \Sigma_T \to \Sigma_T$

since ρ is a congruence

$t_1 \,\rho\, t_1', \ldots\ldots, t_n \,\rho\, t_n'$

$\Rightarrow \quad f_T(t_1, \ldots, t_n) \;\rho\; f_T(t_1', \ldots, t_n')$

__so__

we may define

$$f_{T/E} : \Sigma_{T/E} \times \Sigma_{T/E} \times \ldots \times \Sigma_{T/E} \to \Sigma_{T/E}$$

$$f_{T/E}\left([t_1], [t_2], \ldots [t_n]\right) = \left[f_T(t_1, \ldots, t_n) \right]$$

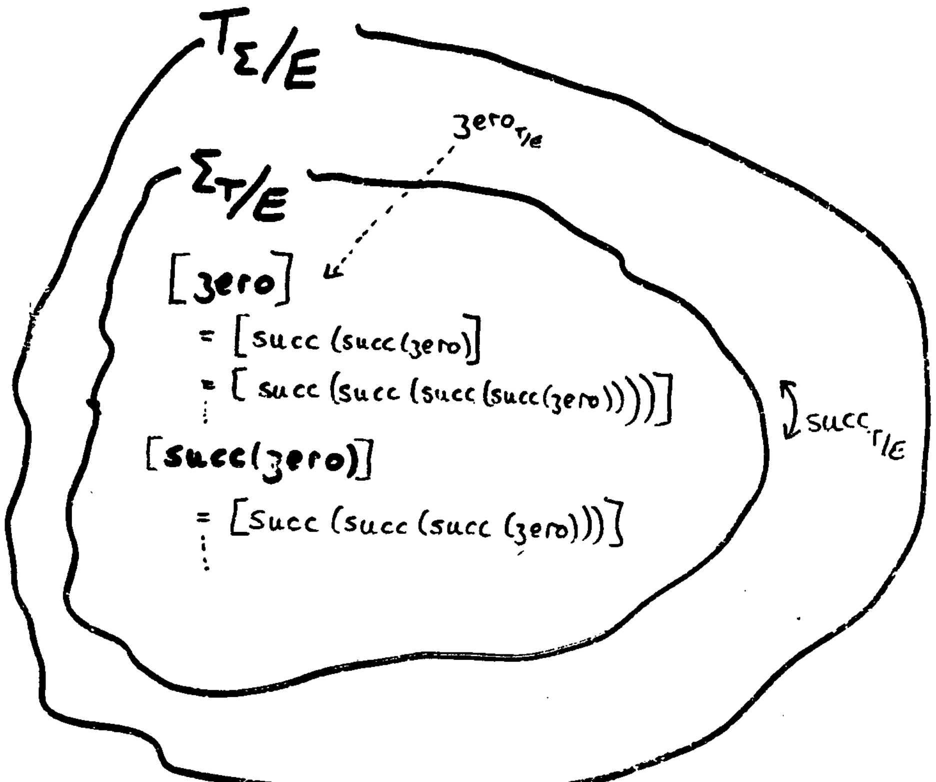
__Fact__ With the operators $f_{T/E}$ as above $\Sigma_{T/E}$ is a (Σ, E) algebra.

The Quotient Term Algebra

Ex

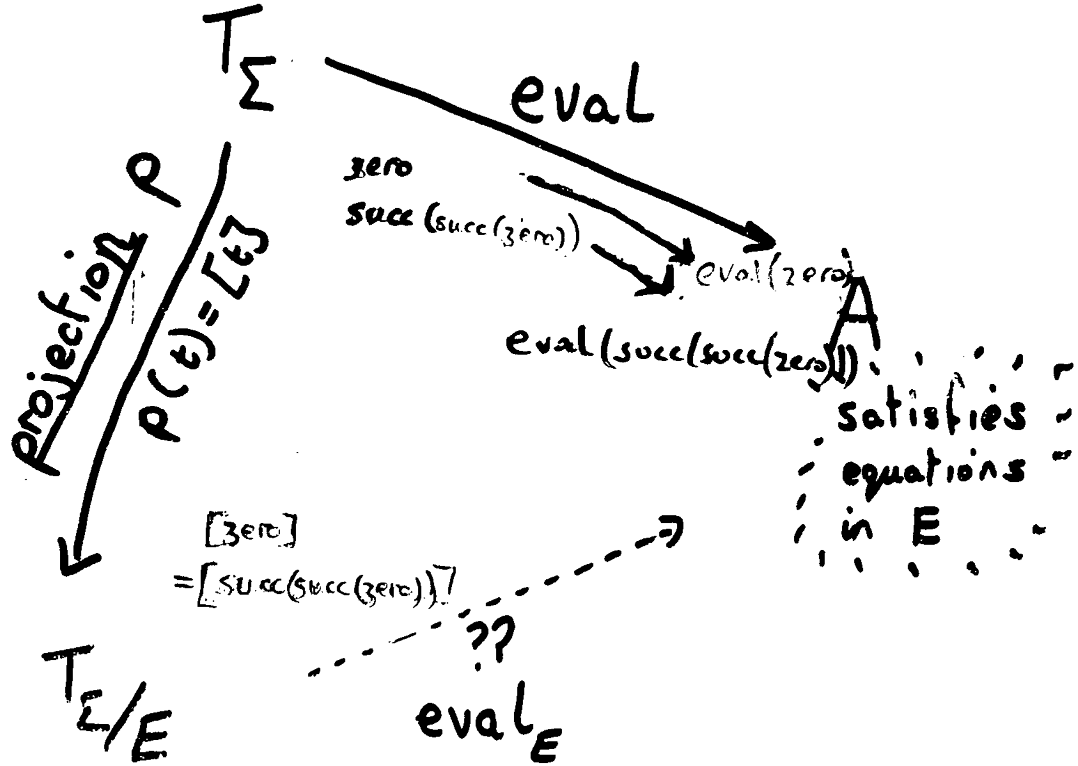
Σ
- sorts nat$_3$
- operators zero : nat$_3$
- succ : nat$_3$, nat$_3$ $\to$ nat$_3$

E
- x : nat
- succ(succ(x)) = x



Examine another model, A, of Σ and see what happens eval over equivalent terms.

definition of eval$_E$



Since A satisfies E

Terms in T_Σ that are identified in $T_{\Sigma/E}$ evaluate to the same element of S_A (in A)

Given a theory presentation

$Th = (\Sigma, E)$

where Σ is a signature

and E is a set of equations

<u>Fact</u>

Let A be any algebra over Th and let

$T_{\Sigma/E}$ denote the quotient term algebra

<u>THEN</u>

There is a unique homomorphism from $T_{\Sigma/E}$ to A

$$eval_E : T_{\Sigma/E} \longrightarrow A$$

<u>INITIAL</u>

<u>Algebra</u>

Ex

let $\Sigma = S, \langle \{a,b,c,d\}, \emptyset, \{\_+\_\}, \emptyset, \emptyset, \emptyset \ldots \rangle$

and let

$$E_1 = \{ ((x+y)+z) = (x+(y+z)) \}$$

$$E_2 = E_1 \cup \{ (x+y) = (y+x) \}$$

$$E_3 = E_2 \cup \{ (x+x) = x \}$$

and let $Th_i = (\Sigma, E_i)$

Which of the previous models for Σ are algebras over Th_1, Th_2, Th_3

What does the homomorphism

$eval/_E$ do in each case?