Homomorphism

Simulation
(Model → Model
(Slide Rule))

Evaluation
(Terms → Model
(Syntax (Language) → Semantics (Meaning))

Assignment
(Free Algebra (Variables) → Term Algebra
Variable $x$ → term $(a+b)+c$

Adding
 VII add XIII XX

Plus
 III plus XIV XIV

Equalities
7 + 13 = 20

Translates
$\text{translate}(7)$ $\text{translate}(13)$ $\text{translate}(20)$
Computer Arithmetic

Simulation of "One Model" by Another

7 \text{ representation of } (7) = 111

4 \text{ representation of } (4) = 100

N, + W, + w

We demand that the operations correspond

\text{representation of } (n+m) = \text{representation of } (n) + w \text{ of } (m)
\[ \log_{10}(1) = 0 \]
\[ \log_{10}(a \times b) = \log_{10}(a) + \log_{10}(b) \]
\[ \log_{10}(\frac{a}{b}) = \log_{10}(a) - \log_{10}(b) \]

\[ \text{GROUP} \]

\[ \text{Homomorphism} \]

Let \( G, * ; e \), \( H, \cdot ; e \) be groups.

Then \( f \) preserves the diadic op.

\[ f(g_i) = e \implies g_i \in \text{identity} \text{ } \text{ and } \text{ } g_i \text{ } \text{ is } \text{ } \text{ preserved } \text{ } \text{ in } \text{ } \text{verses} \]
Machinery/ Homomorphism/ Simulation

sorts: inalph, outalph, state

operators:
next: inalph state -> state
print: state -> outalph

inalph' -> inalph,
\[ \Theta_{\text{inalph}} \]
outalph' -> outalph,
\[ \Theta_{\text{outalph}} \]
state' -> state,
\[ \Theta_{\text{state}} \]

must preserve the operators.
**THEORY**

**HOMOMORPHISM**

Algebra, \( \rightarrow \) Algebra

carrier sets \( \rightarrow \) carrier sets

functions

Preserving operations

hence

Simulation

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**Experiment**

\[ \Sigma = (\text{nat}, \langle K_0, K_1, \emptyset, \emptyset \ldots \rangle) \]

\[ K_0 = \{ \text{zero} \} \]

\[ K_1 = \{ \text{succ} \} \]

Homomorphism?

Let \( \emptyset \) be a homomorphism

\[ \emptyset : T_\Sigma \rightarrow M_1 \]

What can it be???
**Definition (Evaluation of Terms)**

Let \( \Sigma_T \) be the set of terms of a signature \( \Sigma = (s, \langle k_0, k_1, \ldots \rangle) \).

Let \( M \) be a model for \( \Sigma \).

\[ \text{eval} : \Sigma_T \rightarrow S_M \]

is defined inductively as follows:

\[ \text{eval}(n) = n_M \quad \text{for all } n \in K_0 \]

\[ \text{eval}(f(t_1, \ldots, t_r)) = f^M(\text{eval}(t_1), \ldots, \text{eval}(t_r)) \quad \text{for all } r \geq 0 \]

\[ \text{eval}(\text{zero}) = \text{zero}_M = 0 \]

\[ \text{eval}(\text{succ}(\text{zero})) \]

\[ = \text{succ}_M(\text{eval}(\text{zero})) \]

\[ = \text{succ}_M(\text{succ}_M(\text{eval}(\text{zero}))) \]

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\[ \text{eval}(\text{succ}(\text{succ}(\text{zero}))) \]
Fact: $\text{eval}$ is a homomorphism.

Fact: If $M$ is a $\Sigma$ model, then $\text{eval}$ is the only homomorphism from $T_\Sigma$ to $M$.

\[
\text{eval}( ((a+b)+c) ) = (\text{eval}(a+b)) +_M (\text{eval}(c)) = ((a_+_M b_+_M c_+_M) = 9
\]

Examples (eval): $3^1$
Eval

$T_\Sigma$ as before

Lin.

$T_\Sigma$ as before

Examples

eval

eval $( (a + (a + b)) + (a + (c + (d + a))) ) = ?$

eval $(a + \vdash a + c + \vdash d) = ?$
Examples (eval) \[3.17\]

Let \( X = \{x, y, z\} \)
\[ k_0 = \{a, b, c, d\} \]
\[ k_2 = \{+, -\} \]

\( T_\Sigma X \)

What is a homomorphism from \( T_\Sigma X \) to a \( \Sigma \) model \( M \)?

\( T_\Sigma \) as before

eval = ?
A homomorphism $\phi$ from $T_\Sigma(X)$ to a $\Sigma$ model $A$ is called an assignment

Let $(\alpha, \beta)$ be an equation over $\Sigma$ with variables in $X$.

A satisfies $(\alpha, \beta)$ precisely if $\phi(\alpha) = \phi(\beta)$ for all assignments $\phi : T_\Sigma(X) \to A$.

Evaluate $\phi((x * x))$ and $\phi(x)$. Is whatever assignments are made to the variables $\alpha$ & $\beta$ will evaluate to equal values.
Quotient Term Algebra

Given a Theory Presentation \((\Sigma, E)\)

Let \(t_1 \sim t_2 \), \(t_1, t_2 \in \Sigma_T\)

if there is an assignment

\[ \emptyset : T_\Sigma(X) \rightarrow \Sigma \]

s.t. \(\emptyset (\alpha) = t_1\), \(\emptyset (\beta) = t_2\)

and \((\alpha, \beta) \in E\)
Example

\( K_0 = \{ \text{zero} \} \)
\( K_1 = \{ \text{succ} \} \)

\[
\text{succ}(\text{succ}(x)) = x
\]

\( X = \{ x \} \)

\[
\text{proof:}\quad \frac{p}{q} = \frac{r}{s} \text{ if } ps = qr
\]

\[
\begin{align*}
\text{(i)} & \quad \frac{p}{q} + \frac{r}{s} = \left(\frac{ps + qr}{qs}\right) \\
\text{(ii)} & \quad \frac{p}{q} + \frac{r}{s} = \left(\frac{pq + rs}{qs}\right)
\end{align*}
\]

\[
\phi_1(x) = \text{zero} \quad \Rightarrow \quad \text{succ(\text{succ}(\text{zero}))} \n
\phi_2(x) = \text{succ(\text{succ}(\text{zero}))} \quad \Rightarrow \quad \text{succ(\text{succ}(\text{succ}(\text{zero})))}
\]
Let $\rho$ be the smallest congruence on $T_\Sigma$ containing $\sim$

For each $f \in K_n$, we have $f_t : \Sigma_t \times \ldots \times E_t \to \Sigma_t$

since $\rho$ is a congruence $t_1, \rho t_1', \ldots, t_n, \rho t_n' \Rightarrow f_t(t_1, \ldots, t_n) \rho f_t(t_1', \ldots, t_n')$

So we may define $f_{T/E} : \Sigma_{T/E} \times \Sigma_{T/E} \times \ldots \Sigma_{T/E} \to \Sigma_{T/E}$

$\left[ f_{T/E} \left( [t_1], [t_2], \ldots, [t_n] \right) \right] = \left[ f_t(t_1, \ldots, t_n) \right]$

Fact With the operators $f_{T/E}$ as above $\Sigma_{T/E}$ is a $(\Sigma, E)$ algebra.

The Quotient Term Algebra
\[
\Sigma = \{\text{sorts } \text{nat}, \text{ operators } \text{zero} : \text{nat}, \text{ succ} : \text{nat}, \text{nat} \rightarrow \text{nat} , \text{x} : \text{nat} \}
\]

\[
\Sigma = \{\text{succ(succ(x)) = x} \}
\]

\[
\Sigma/E = \{\text{zero, succ(zero)} \}
\]

\[
\Sigma/E = \{\text{succ(succ(zero))} \}
\]

\[
T_{\Sigma/E} \rightarrow \text{eval}_{\Sigma/E} \]

\[
\rho(\epsilon) = [\epsilon] \]

\[
\text{eval(\epsilon)} = [\text{succ(succ(\epsilon))}] \]

\[
\text{Since A satisfies E, terms in } T_{\Sigma} \text{ that are identified in } T_{\Sigma/E} \text{ evaluate to the same element of } S_A \text{ in A} \]
Given a theory presentation
\[ \mathcal{Th} = (\Sigma, E) \]
where \( \Sigma \) is a signature
and \( E \) is a set of equations

\textbf{Fact}
Let \( A \) be any algebra
over \( \mathcal{Th} \) and let
\[ T_{\Sigma/E} \]
denote the quotient
term algebra

\textbf{THEN}
There is a unique homomorphism
from \( T_{\Sigma/E} \) to \( A \)

\[ \text{eval}_E : T_{\Sigma/E} \rightarrow A \]
Let \( \Sigma = S, \langle \varepsilon, a, b, c, d, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle \)

and let

\[
E_1 = \{ ((x+y)+z) = (x+(y+z)) \}
\]

\[
E_2 = E_1 \cup \{ (x+y) = (y+x) \}
\]

\[
E_3 = E_2 \cup \{ (x+x) = x \}
\]

and let \( \text{Th}_i = (\Sigma, E_i) \)

Which of the previous models for \( \Sigma \) are algebras over \( \text{Th}_1, \text{Th}_2, \text{Th}_3 \)

What does the homomorphism \( \text{eval}_E \) do in each case?