Background

- Syntax as term algebras over a signature
- Denotational Semantics as an interpretation
  - justified by behavioural criteria
  - characterised by equations

Outline

1. Why extra structure is required
2. Brief look at modifications to the underlying theory
RESULT

\[ \text{Eval}_{CIE} ( ) < \text{Eval}_{CIE} ( ) \implies t \in \mathcal{P} \leq \]

THE PROOF SYSTEM IS COMPLETE
with the particular interpretation \( CIE \)

MORAL: CHOOSE A SEMANTICS ISOMORPHIC
TO \( CIE \) FOR SOME \( E \).
- you will then have a "complete transformation system" for semantic equality

TO OBTAIN AN EFFECTIVE SYSTEM REPLACE
\( \omega \)-Induction with

RECUSION INDUCTION:

\[
\frac{t [u/x] \leq u}{\text{rec}_x.t \leq u}
\]
Example 1: Structure Induced by Behavioural Criteria

\[ P \leq_{\text{tot}} Q \]

For every test \( e \), if \( P \) must satisfy \( e \) implies \( Q \) must satisfy \( e \), then \( Q \) is at least as deterministic as \( P \).

- To reflect, we need interpretation, where the carrier sets come equipped with a \( \leq \).
- Hoped for:

\[ \text{eval}_I (P) \leq \text{eval}_I (Q) \quad \text{if} \quad P \leq_{\text{tot}} Q \]
\[ \Sigma - \text{a signature} \]

A \[ \Sigma \text{- po interpretation consists of} \]
- carrier set \( \Delta \)
- partial-order \( \leq_C \) on \( \Delta \)

\[ t \leq t \]
\[ t \leq u, u \leq r \implies t \leq r \]
\[ t \leq u, u \leq s \implies t \leq s \]

- a function \( b_a \) for each symbol \( a \) in \( \Sigma \) which is \text{monotonic}

\[ a \leq a b \implies b(a) \leq b(b) \]

**Example**

\[ \Sigma \text{ is } 0 \text{ Succ} \]
- carrier set \( N \) - natural numbers
- partial order \( \leq_N \) - numerically less than or equal to
- functions - the usual

\[ 0 \text{ } N \text{ number } 0 \]
\[ \text{Succ}_N (n) = n + 1 \]
EXAMPLE \[ \Sigma \text{ is } \text{NIL, } \alpha, + \]
- carry out PS - non-empty prefix-closed subsets of \( \text{Act} \)
- partial order \( \leq_{ps} : S \leq_{ps} T \) if \( S \subseteq T \)
- functions NIL\(_{ps} \), \( \alpha_{ps} \), \( +_{ps} \) as before

\[ \Sigma \text{- po homomorphisms} \]

if \( \text{A}, \text{B} \) are \( \Sigma \text{- po algebras}, h : \text{A} \rightarrow \text{B} \)
is a \( \Sigma \text{- po homomorphism} \) if:

i) \( h \) is a \( \Sigma \text{- homomorphism} \)
    - preserves the \( \Sigma \text{- structure} \)

ii) \( h \) is monotone:
    a \( \leq_{A} a' \) implies \( h(a) \leq_{B} h(a') \)
    - preserves the partial-order structure.
INITIALITY

If $I$ is an interpretation of $\Sigma$, it is also a $\Sigma$-po-interpretation:

we only need a partial-order on the carrier

$a \leq I b$ if $a = b$

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RESULT: THE WORD ALGEBRA $T_\Sigma$ is a

po-interpretation

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If $I$ is a $\Sigma$-po interpretation, it is also an interpretation for $\Sigma$

RESULT: Eavl$_I : T_\Sigma \rightarrow I$ is a $\Sigma$-po homomorphism

$T_\Sigma$ is the initial $\Sigma$-po interpretation

RESULT: USE $T_\Sigma$ AS SYNTAX

- USE $\Sigma$-po interpretations as denotation semantics

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- Every program will have a unique
  behaviorally-induced semantics
INEQUATIONAL THEORIES

INEQUATIONS:

N satisfies \( 0 \leq x \)

\( \mathbb{P} \mathbb{S} \) satisfies \( x \leq x + y \)

RESULT: For any set of inequations \( E \)

There is a "least" \( \Sigma \mathbb{P} \mathbb{O} \) interpretation which satisfies \( E \), \( \Sigma_{/E} \)

- there is a unique \( \Sigma \mathbb{P} \mathbb{O} \) homomorphism \( \text{eval}_{I} : \Sigma_{/E} \rightarrow I \) for every \( I \) which satisfies \( E \).

- \( \Sigma_{/E} \) is determined by the proof system:

\[
\begin{align*}
t < t & \quad t < u, u < r \quad \Rightarrow \quad t < r \\
t < u & \quad t < u \quad \Rightarrow \quad \mathbb{P}(t) < \mathbb{P}(u) \\
\mathbb{P}p < \mathbb{P}u & \quad \text{for every } t < u \text{ in } E
\end{align*}
\]
EXAMPLE PS is determined by:

\[ x + x = x \quad x + x \leq x \]
\[ x + y = y + x \quad x + y \leq y + x \]
\[ x + (y + z) = (x + y) + z \quad x + \text{NIL} \leq x \]
\[ a(x + y) = ax + ay \quad a(x+y) \leq ax + ay \]
\[ x \leq x+y \]

RESULT: WE CAN CHOOSE PARTICULAR DENOTATIONAL SEMANTICS WHICH IS CHARACTERISED BY A SET OF INEQUALATIONS AND THEREBY GET A COMPLETE PROOF SYSTEM FOR SEMANTIC EQUALITY.

RESULT: THERE IS NO CHANGE TO THE ORIGINAL APPROACH BASED ON SIMPLE INTERPRETATIONS.
EXAMPLE TWO OF EXTRA STRUCTURE

Interpreting recursive or infinite terms

Example

\[ x \equiv a(b\text{null} + cdx) \]

Example

\[ x \equiv (abx)\parallel cx \]

Problem 1: It is difficult to consider the syntax as a word algebra

\[ \text{recx. } a(b\text{null} + cdx) \]
\[ \text{recx. } (abx)\parallel cx \]
NEW SYNTAX: RECURSIVE TERMS
- add variables + constructor recx.
- terms in REC2 - recursive terms
\[ t ::= \text{nil} \mid \text{at} \mid t + u \mid x \mid \text{recx}.t \]

PROBLEM 2

HOW DO WE INTERPRET
\[ \text{recx}. a \cdot x + b \]?

intuition: interpret as solution to
\[ x = a_I(x) + b_I \]

BUT: MAYBE NO SOLUTION

e.g. in PS

MAYBE MANY SOLUTIONS

e.g. \[ x = a_I + x \]

has solutions
\[ \{e, a\}, \{e, a, b\}, \{e, a, b, c\}, \ldots \]

in PS

- ADD STRUCTURE TO INTERPRETATIONS TO
ENSURE EVERY EQUATION HAS A "LEAST" SOLUTION
RE-EXAMINE SYNTAX

Recall, $ax + b \text{nil}$ is a finite representation of an infinite word.

This infinite word can be approximated by an infinite sequence of finite words.

First approx.

Second approx.

Third approx.
\( \Sigma \)-complete partial order interpretation

**Intuition:** to ensure the interpretation of the sequence of limits approximations to the term \( \pi \) has a limit.

**Definition:**

- \( \text{cpo} \) is a set \( \Pi \) equipped with a partial order \( \leq \) which satisfies:
  - there is a least element \( \bot \) under \( \leq \) to interpret?
- every chain \( C : c_0 \leq c_1 \leq \ldots \) has a least upper bound \( \sup C \)

**Definition**

\( \Sigma \)-cpo interpretation is an interpretation \( I \) for \( \Sigma \) where the carrier is a \( \text{cpo} \) and the functions are continuous:

\[ I(\sup C) = \bigvee \{ I(c) \mid c \in C \} \]
- If we choose a $\Sigma$-crp as an interpretation then every term in $\text{REC}_\Sigma$ has a meaning.

RESULT

$$\text{eval}_I : \text{REC}_\Sigma \to I \quad \exists$$

It is the unique $\Sigma$-homomorphism which satisfies

$$\text{eval}_I(p) = \bigvee \{ \text{eval}_I(d) \mid d \in \text{App}(p) \}$$

- The meaning of a program is structurally induced.

- The meaning of an infinite term is determined by the meaning of its finite approximations.

- The meaning of exact is the least solution in $I$ of the equation

$$x = t$$
- must be a $\Sigma$-cpo
- must be fully-abstract w.r.t. $\leq_{st}$
- determined by a set of (in)equations

RESULT: FOR ANY SET OF EQUATIONS $E$ THERE A "LEAST" $\Sigma$-cpo WHICH SATISFIES THEM

$CIE$ - initial $\Sigma$-cpo in the class which satisfies the equations $E$

$CIE$ constructed by factoring the infinite $\Sigma$-words with the equations $E$

Given a $\Sigma$-cpo $I$ which satisfies the equations $E$, there is a unique $\Sigma$-cpo homomorphism

$$\text{eval}_I^E : CIE \rightarrow I$$
EQUATIONALLY CHARACTERISED
DENOTATIONAL SEMANTICS

\[ \text{if } t \leq u \text{ and } u \leq t \text{ then } t = u \]

\[ \text{if } t \leq u \text{ and } p(t) \leq p(u) \text{ then } tp \leq up \]

\[ t \leq u \text{ for every } t, u \text{ in } E \]

\[ \text{if } d \leq u \text{ for every } d \in \text{App}(t) \text{ then } t \leq u \]

W-INDUCTION

\[ \text{if } E \vdash t \leq u \text{ then } t \leq u \text{ can be derived using these rules} \]

RESULT:

IF \( I \) is a \( \Sigma \)-corp which satisfies \( E \) then \( E \vdash t \leq u \) implies \( \text{eval}_I(t) \leq \text{eval}(u) \)

SOUNDNESS OF THE PROOF SYSTEM with any interpretation which satisfies the inequalities.
RESULT

\[ \text{eval}^{CIE} (x) \leq \text{eval}^{CIE} (y) \text{ implies } \exists p \leq y \]

THE PROOF SYSTEM IS COMPLETE WITH THIS PARTICULAR INTERPRETATION \( CIE \)

MORAL: CHOOSE A SEMANTICS ISOMORPHIC TO \( CIE \) FOR SOME \( E \).

you will then have a "complete transformation system" for semantic equality

TO OBTAIN AN EFFECTIVE SYSTEM REPLACE

\( \text{\textsc{w}} \)-INDUCTION WITH

(recursion induction)

\[
\frac{t[x/u] \leq u}{\text{rec} \alpha.t \leq u}
\]
EXAMPLE APPLICATION - theoretical CSP

SYNTAX: recursive terms over a signature $\Sigma^2$

$t ::= a | t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9 x | \text{rec} . t$

SEMANTICS: $\Sigma^2$-cpo

- e.g. Acceptance Trees
- Bounded Refusal Sets

Result: every CSP term has a unique meaning as an Acceptance Tree

BEHAVIOURAL CHARACTERISATION:

Acceptance Trees are fully-abstract wrt $\leq_{\text{st}}$

$\text{eval}_{\text{AT}}(P) \leq \text{eval}_{\text{AT}}(Q)$ if and only if $P \leq_{\text{st}} Q$

COMPLETE TRANSFORMATION SYSTEM

$\text{WIE}(\mathcal{A}, \mathcal{R})$ is sound + complete wrt Acceptance Trees

$\uparrow$

$\text{W-Induction + Equational Reasoning}$
BCS FACS CHRISTMAS WORKSHOP 1987
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