ALGEBRAS WITH STRUCTURE

<u>background</u>

- SYNTAX as term algebras over a signature
- Denotational Semantics as an interpretation of
  - justified by behavioural criteria
  - characterised by equations

OUTLINE

1. WHY EXTRA STRUCTURE IS REQUIRED

2. BRIEF LOOK AT MODIFICATIONS TO THE UNDERLYING THEORY

DENOTATIONAL SEMANTICS

RESULT

$\mathrm{Eval}_{CI_E}(\ ) < \mathrm{eval}_{CI_E}(\ )$ implies $\vdash^{\omega}_{E} P \leq$

THE PROOF SYSTEM IS COMPLETE wrt this particular interpretation CI_E

MORAL: CHOOSE A SEMANTICS ISOMORPHIC TO CI_E FOR SOME E.
- you will then have a "complete transformation system" for semantic equality

TO OBTAIN AN EFFECTIVE SYSTEM REPLACE ω-Induction with

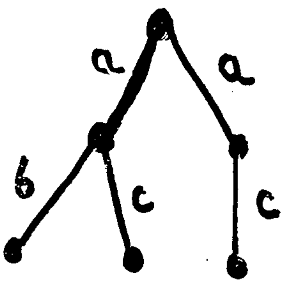
RECURSION INDUCTION :
$$\frac{t[u/x] \leq u}{\mathrm{rec}\,x.t \leq u}$$

EXAMPLE 1 STRUCTURE INDUCED BY BEHAVIOURAL CRITERIA

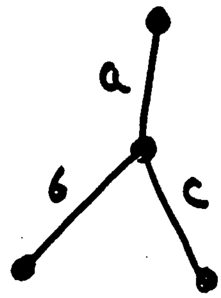
$p \leq_{test} q$ if for every test e

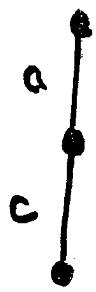
p must satisfy e implies q must satisfy e

q is at least as deterministic as p

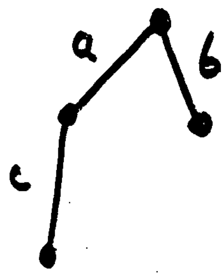


$\leq_{test}$





$\not\leq_{test}$



- TO REFLECT WE NEED INTERPRETATIONS WHERE THE CARRIER SETS COME EQUIPED WITH A $\leq$

- HOPED FOR:

$$eval_I(p) \leq eval_I(q) \quad \text{iff} \quad p \leq_{test} q$$

Σ - a signature

A **Σ-po interpretation** consists of

- carrier set A
- partial-order $\leq_A$ on A

$t \leq t$

$t \leq u,\ u \leq r$ implies $t \leq$

$t \leq u,\ u \leq t$ implies $t=$

- a function f_A for each symbol f in Σ which is **monotonic**

$$a \leq_A b \quad \text{implies} \quad f(a) \leq_A f(b)$$

EXAMPLE Σ is 0 Succ

- carrier set N - natural numbers
- partial order $\leq_N$ - numerically less than or equal to
- functions - the usual

0_N number 0

$Succ_N(n) = n+1$

EXAMPLE Σ is NIL, a, $+$

- carrier set PS - non-empty prefix-closed subsets of Act^*
- partial order $\leq_{PS}$: $S \leq_{PS} T$ if $S \subseteq T$
  subset inclusion
- functions NIL_{PS}, a_{PS}, $+_{PS}$ as before

Σ- po homomorphisms :

if A, B are Σ- po algebras, $h : A \longrightarrow B$ is a <u>Σ- po homomorphism</u> if:

i) h is a Σ- homomorphism
 - preserves the Σ- structure

ii) h is monotonic :

$$a \leq_A a' \quad \text{implies} \quad h(a) \leq_B h(a')$$

 - preserves the partial-order structure.

INITIALITY

If I is an interpretation of Σ it is also a Σ po-interpretation :

we only need a partial-order on the carrier

$a \leq_I b$ if $a = b$

- degenerate partial-order

RESULT : THE WORD ALGEBRA T_Σ is a po-interpretation

If I is a Σ-po interpretation it is also an interpretation for Σ

RESULT : $Eval_I : T_\Sigma \to I$ is a Σ-po homomorphism

T_Σ is the initial Σ-po interpretation

RESULT : - USE T_Σ AS SYNTAX

- USE Σ-po interpretations as denotation semantics

- Every program will have a unique structurally-induced mea...

INEQUATIONAL THEORIES

INEQUATIONS:

N satisfies $0 \leq x$

PS satisfies $x \leq x + y$

RESULT: FOR ANY SET OF INEQUATIONS E THERE IS A "LEAST" Σ-po INTERPRETATION WHICH SATISFIES E, $T_{\Sigma/E}$

- there is a unique Σ-po homomorphism $eval_I^E : T_{\Sigma/E} \longrightarrow I$ for any I which satisfies E.

- $T_{\Sigma/E}$ is determined by the proof system

$$\frac{}{t < t} \qquad\qquad \frac{t < u, \; u < r}{t < r}$$

$$\frac{t < u}{t\rho < u\rho} \qquad\qquad \frac{t < u}{f(t) < f(u)}$$

$$\frac{}{t < u} \quad \text{for every } t < u \text{ in E}$$

EXAMPLE PS is determined by:

$x + x = x$ — $x + x \leq x$, $x \leq x + x$

$x + y = y + x$ — $x + y \leq y + x$, $y + x \leq x + y$

$x + (y + z) = (x + y) + z$ —

$x + NIL < x$

$a(x + y) = ax + ay$ — $a(x+y) \leq ax + \ldots$, $ax + ay \leq a(x\ldots$

$x \leq x + y$

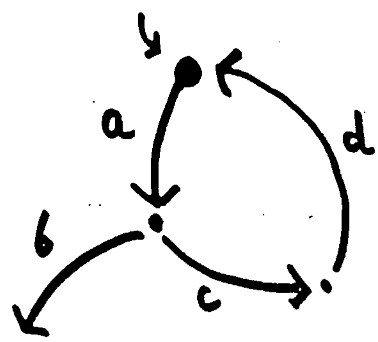
RESULT: WE CAN CHOOSE PARTICULAR DENOTATIONAL SEMANTICS WHICH IS CHARACTERISED BY A SET OF INEQUATIONS + THEREBY GET A COMPLETE PROOF SYSTEM FOR SEMANTIC EQUALITY

RESULT: THERE IS NO CHANGE TO THE ORIGINAL APPROACH BASED ON SIMPLE INTERPRETATIONS

EXAMPLE TWO OF EXTRA STRUCTURE

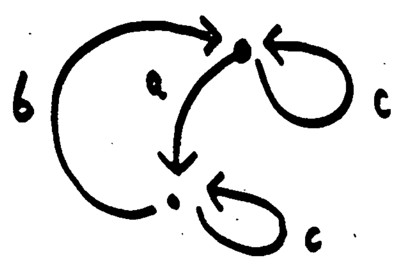
Interpreting recursive or infinite terms

Example $\quad x \Leftarrow a(b\,\mathrm{NIL} + cdx)$



Example $\quad x \Leftarrow (abx) \| cx$

run in parallel!



PROBLEM 1 : IT IS DIFFICULT TO CONSIDER THE SYNTAX AS A WORD ALGEBRA

$\mathrm{rec}\,x.\ a(b\,\mathrm{NIL} + cdx)$

$\mathrm{rec}\,x.\ (abx)\|cx$

NEW SYNTAX : RECURSIVE TERMS

- add variables + constructor $rec\,x.\_$

terms in REC_Σ - recursive terms

$$t ::= \text{NIL} \mid a\,t \mid t + u \mid x \mid rec\,x.t$$

PROBLEM 2 HOW DO WE INTERPRET

$rec\,x.\,a\,x + b$?

intuition : interpret as solution to

$$x = a_I(x) +_I b_I$$

BUT : MAYBE NO SOLUTION

e.g in PS

MAYBE MANY SOLUTIONS

e.g. $x = a_I + x$

has solutions

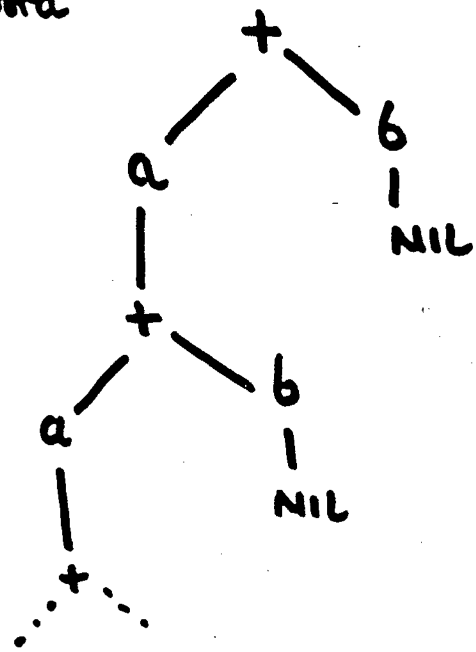
$\{\varepsilon, a\}, \{\varepsilon, a, b\}, \{\varepsilon, a, b, bc\}, \ldots$

in PS

- ADD STRUCTURE TO INTERPRETATIONS TO ENSURE EVERY EQUATION HAS A "LEAST" SOLUTIO

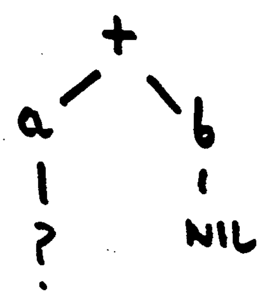
RE-EXAMINE SYNTAX

rec x. $ax + b$NIL is a finite representation of an infinite word

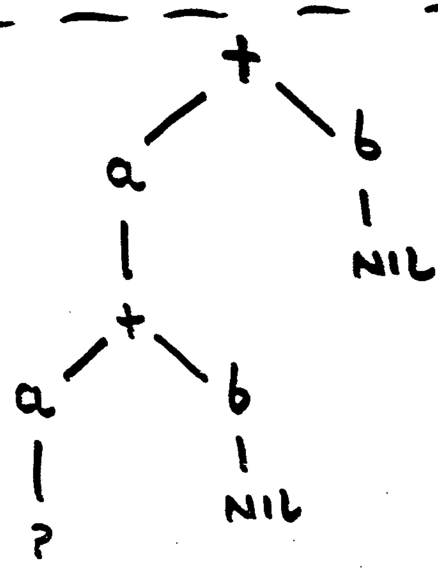


THIS INFINITE WORD CAN BE APPROXIMATED BY AN INFINITE SEQUENCE OF FINITE WORDS

? = unknown   first approx



second approx



third approx

Σ- complete partial order interpretation

intuition : to ensure the interpretation of the sequence of finite approximations to the term p has a limit.

Definition :

- *cpo* is a set A equipped with a partial order $\leq$ which satisfies
  i) there is a least element $\perp_A$ wrt $\leq$ to interpret ?
- every chain C : $c_0 \leq c_1 \leq \ldots\ldots$ has a least upper bound $\vee C$

Definition

Σ- cpo interpretation is an interpretation I for Σ where the carrier is a cpo and the functions are continuous:

$$f(\vee C) = \vee\{f(c),\ c \in C\}$$

- IF WE CHOOSE A Σ-cpo as an interpretation then every term in REC_Σ has a meaning

RESULT

$$eval_I : REC_\Sigma \longrightarrow I \qquad \text{exists}$$

It is the unique Σ-homomorphism which satisfies

$$eval_I(p) = \bigvee \{ eval_I(d),\ d \in App(p) \}$$

- the meaning of a program is structurally induced

- the meaning of an infinite term is determined by the meaning of its finite approximations

- the meaning of rec $x.t$ is the least solution in I of the equation

$$x = t$$

- must be a Σ-cpo
- must be fully-abstract wrt. $\leq_{tst}$
- determined by a set of (in)equations

<u>RESULT</u> : FOR ANY SET OF EQUATIONS E THERE A "LEAST" Σ-cpo WHICH SATISFIES THEM

CI_E - initial Σ-cpo in the class which satisfies the equations E

CI_E constructed by factoring the infinite Σ-words with the equations E

given a Σ-cpo I which satisfies the equations E there is a unique Σ-cpo homomorphism

$$eval_I^E : CI_E \longrightarrow I$$

EQUATIONALLY CHARACTERISED DENOTATIONAL SEMANTICS

$$\frac{}{t \le t} \qquad \frac{t \le u}{u \le t} \qquad \frac{t \le u,\ u \le r}{t \le r}$$

$$\frac{t \le u}{f(t) \le f(u)} \qquad \frac{t \le u}{t\rho \le u\rho} \qquad \frac{}{t \le u}$$

for every $t \le u$ in E

$$\frac{d \le u,\ \text{for every } d \in App(t)}{t \le u}$$

ω-INDUCTION

$\vdash^{\omega}_{E} t \le u$ means $t \le u$ can be derived using these rules

RESULT :

IF I is a Σ-cpo which satisfies E then $\vdash^{\omega}_{E} t \le u$ implies $eval_I(t) \le eval(u)$

- SOUNDESS OF THE PROOF SYSTEM wrt any interpretation which satisfies the inequations

DENOTATIONAL SEMANTICS

RESULT

$\text{Eval}_{CI_E}(\ ) < \text{Eval}_{CI_E}(\ )$ implies $\vdash^{\omega}_{E} p \leq q$

THE PROOF SYSTEM IS COMPLETE
wrt this particular interpretation CI_E

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SYNTAX : recursive terms over a signature Σ^2

$$t ::= stop \mid a \to t \mid t \sqcap u \mid t \,\square\, u \mid t \| u \mid$$
$$t - a \mid x \mid rec\,x.t$$

SEMANTICS : Σ^2- cpo

e.g. Acceptance Trees

Bounded Refusal Sets

Result: every CSP term has a unique meaning as an Acceptance Tree

BEHAVIOURAL CHARACTERISATION:

Acceptance Trees are fully-abstract wrt $\leq_{test}$

$eval_{AT}(p) \leq eval_{AT}(q)$ if and only if $p \leq_{test} q$

COMPLETE TRANSFORMATION SYSTEM

$\omega ⟦E⟧ (Ax)$ is sound + complete wrt Acceptance Trees

$\uparrow$

ω-induction + equational reasoning

BCS FACS CHRISTMAS WORKSHOP 1987
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J Bicarregui	RAL
David Blyth	Incord
David Bosomworth	Racal
R Bruynooghe	
Martin Bush	South Bank Poly
R Carsley	PCL
J H Cheng	IST
Derek Coleman	HP
Stephen Colwill	Praxis
John Cooke	LUT
Jim Cunningham	IC
John Dawes	ICL
Tim Denver	Praxis
Jeremy Dick	RAL
David Ellis	Praxis
Paul Farrow	Program Validation Ltd
C George	STL
Abida Ghoni	PRG Oxford
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Christopher Holt	Newcastle Univ
H P Houghton	Wolverhampton Poly
S Hughes	STL
Mike Johnson	BT
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Neil Lancaster	Praxis
Ming Lee	TSL Communications
Sheelagh LLoyd	Racal
Noel Lobo	Racal
Matthew Love	Sheffield Poly
Robert Low	Coventry Poly
Catherine Lowe	Racal Research
J Lynch	BT
Ursula Martin	RH & BNC
S R Matthews	Hatfield Poly
Richard Mitchell	Hatfield Poly
Margaret Myers	South Bank Poly
Gordon Nichols	Hatfield Poly
C O'Halloran	RSRE
J Peacham	NE London Poly
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David Smith	Hatfield Poly
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Roger Stone	LUT
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Imperial College staff

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Rachel Harrison
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