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Matrices of Sets

- Nature and structure
- Origin and provenance
- Operations with Matrices of Sets
- Scanning Use Cases
- Next Steps

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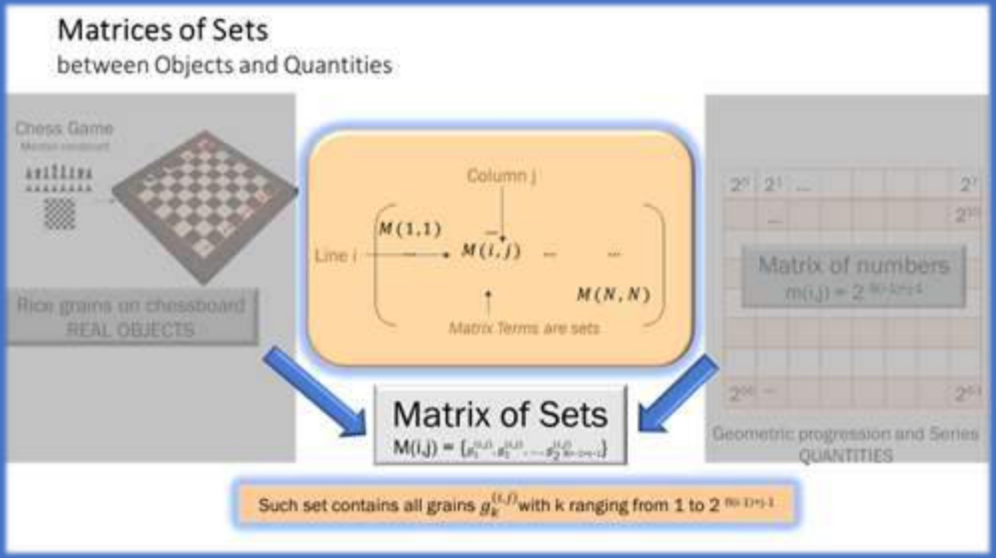
Matrices of Sets: Seminars in 2021

4th March			Applied Mathematics, Physics, Data Science
1st April			Multimodal Container Transport
19th April			Geographic Information Analysis & Visualisation
6th May			Computer Science, Data Science
23rd Sept.	British Computing Society		**Computer Science, Data Science,** Formal Aspects of Computing Science SG

Matrices of Sets:

-Nature and structure



Matrix structure

- A double index container of information

  The classical matrix is M with term m_{ij} at line i and column j: *table of numbers*

  -Matrices are ubiquitous in any science and engineering, increasingly so as data availability and appetite for it increases, for instance in AI.
- **Replace term at position i,j by a set M_{ij}**

Set nature

- **Flexibility of sets**: sets describe whatever is needed

  see set defined in comprehension = concise expression of properties of elements, or set defined in extension = enumerating elements
- *Numbers* are just a summary of "things", vectors and linear spaces can be defined efficiently only when some homogeneity is imposed (state vectors, parameters of a system, defined observed system, etc). Obviously: quantitative modelling!
- **Programmability of sets**: history going back to LISP at least, where the central object was a list (a set is a list from which the order of enumeration has been removed)
- **Data sets** always more important in Digital and AI era

Origin and Provenance

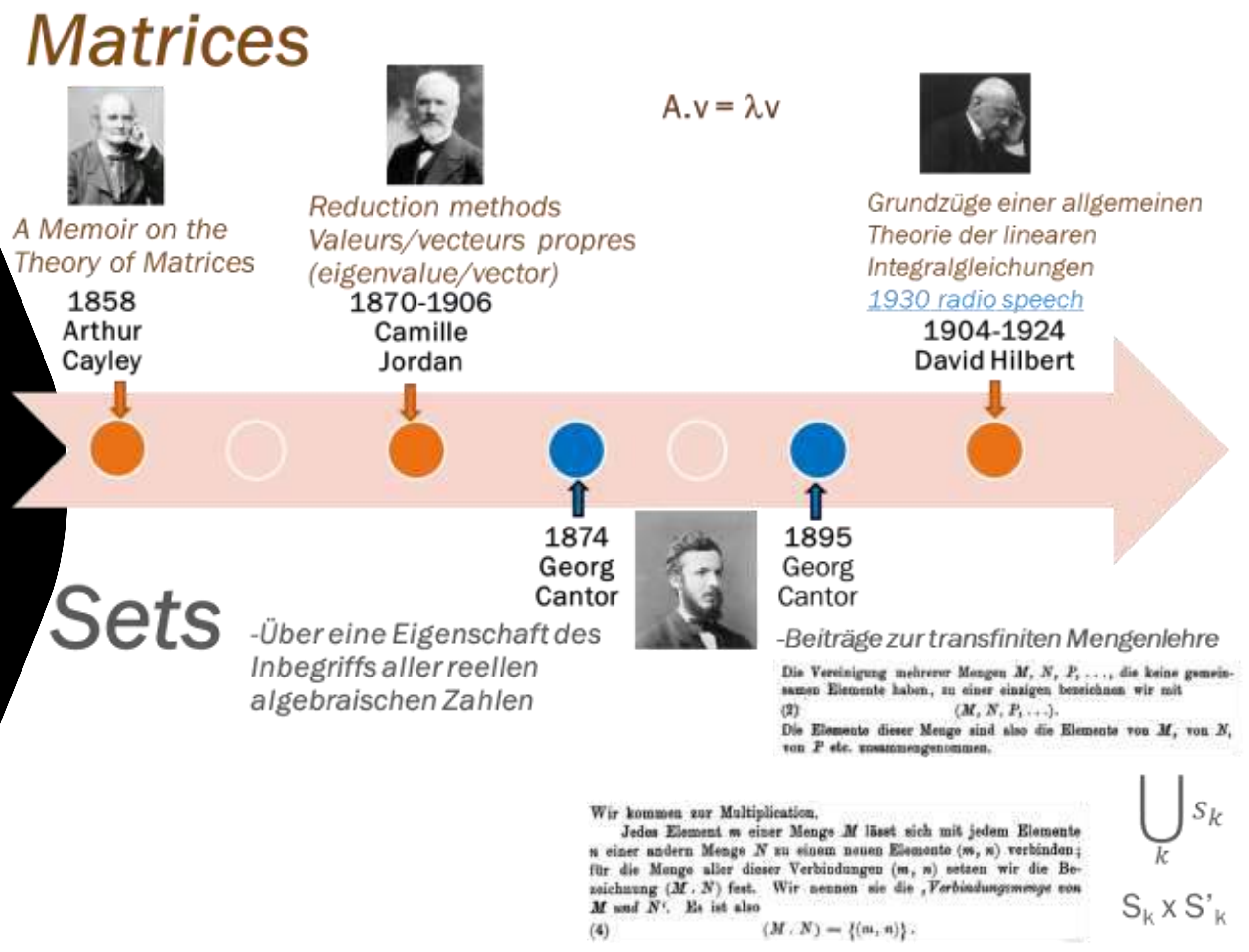
1850-1930 Matrices develop

from solving linear equations, and later linear differential equations

-today: matrices are part of the curriculum of the European Baccalaureate; used in every area of science and engineering

1874-1895 Set Theory introduced, by Georg Cantor

-today: after being in the curriculum of 11 years old students (FR), it is now postponed to Baccalaureate level at latest



Operations with Matrices of Sets

Matrix

M defined by numbers $m_{i,j}$

Matrix product M.M' defined by C with number terms $c_{i,j}$

$$c_{ij} = \sum_k m_{ik} m'_{kj}$$

Sum of number products

NB for numbers p.q =0 ⇒ p=0 or q=0

Matrix of Sets

M defined by sets $M_{i,j}$

Matrix product MxM' defined by C with set terms $C_{i,j}$

$$C_{ij} = \bigcup_k M_{ik} \text{ x } M'_{kj}$$

Reunion of Cartesian set products

NB for sets AxB = ∅ ⇒ A = ∅ or B= ∅

Reminder

- AxB =∅ ⇒ there exist no (a,b) such that a ∈A and b∈ B
⇒ [not (∃a ∈A)] v [not(∃b ∈B)] ⇒ A = ∅ or B= ∅

Spectral Theory: eigenvector, eigenvalue

Matrix

Vector v and scalar λ such that

$M.v = \lambda\, v$

$M.v = \mathbb{1}(\lambda).v$

rewritten with $\mathbb{1}(\lambda)$ the diagonal matrix with λ on the diagonal and zero elsewhere

Matrix of Sets

Vector of sets V and set Λ such that

$M.V = \mathbb{1}(\Lambda).V$

where $\mathbb{1}(\Lambda)$ is the diagonal matrix of sets with Λ on the diagonal and $\varnothing$ elsewhere

2x2 Matrix of Sets illustration, for a "hands-on feel"

Take

$M = \begin{pmatrix} A & B \\ \emptyset & C \end{pmatrix}, \quad V = \begin{pmatrix} X \\ Y \end{pmatrix}$

The eigenvalue, eigenvector property is

$M x V = \mathbb{1}(\Lambda) \times V$

or

(i) $A \times X \ \cup \ B \times Y = \Lambda \times X$

$C \times Y = \Lambda \times Y$

Hence $C = \Lambda$ if $Y \neq \emptyset$

(i) requires that

$B \times Y \subset \Lambda \times X$

and

$A \times X \subset \Lambda \times X$

Hence

$B \subset \Lambda$, $A \subset \Lambda$, $Y \subset X$

From $Y \subset X$,

and

$\Lambda \times X = A \times X \ \cup \ B \times Y$

we get

$\Lambda \times X \subset (A \cup B) \times X$

Hence

(ii) $\Lambda \subset A \cup B$

However from $B \subset \Lambda$, $A \subset \Lambda$, we have

(iii) $A \cup B \subset \Lambda$

Hence

(iv) $A \cup B = \Lambda$

Decompose X which contains Y, into Y and its complement in X, T:

$X = Y \cup T$, with $Y \cap T = \emptyset$

Then (i) becomes

$\Lambda \times X = A \times X \ \cup B \times Y = (A \cup B) \times Y \ \cup \ A \times T$

And

$\Lambda \times Y \ \cup \Lambda \times T = \Lambda \times Y \ \cup \ A \times T$

Projecting onto T which has no intersection with Y, $\Lambda = A$

Hence $B \subset A$

We can therefore write :

Theorem

Consider $M = \begin{pmatrix} \Lambda & B \\ \emptyset & \Lambda \end{pmatrix}$

Suppose

(i) $B \subset \Lambda$

(ii) $V = \begin{pmatrix} X \\ Y \end{pmatrix}$ such that $Y \subset X$

then $M x V = \mathbb{1}(\Lambda) x V$

More theorems +1

Theorem on triangular matrices

with increasing lines and constant diagonal

Assume that the Matrix of Sets A is lower triangular, with constant set value Λ along the diagonal,and increasing along each line until the diagonal, which can be written as:

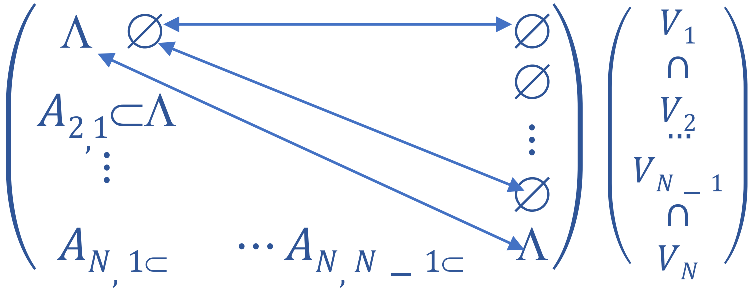
(a) $\forall i \in [1,N], \forall k \in [i+1,N], A_{i,k}=\emptyset$

(b) $\forall i \in [1,N], \forall k \in [1,i-1], A_{i,k} \subset A_{i,(k+1)}$

(c) $\exists\Lambda, \forall i \in [1,N], A_{i,i}=\Lambda$

(d) $\forall k \in [1, N-1], V_k \subset V_{k+1}$

Then, for Matrix of Sets A ,any **V** satisfying (d) and non-empty, is an **eigenvector of sets** for **eigenvalue set** Λ



More theorems +1

Theorem of « constant line sum »

If there exists a set Λ such that that for every i=1, …, N

$$\bigcup_k A_{ik} = \Lambda$$

Then

$$V=\begin{pmatrix} W \\ W \\ \dots \\ W \end{pmatrix}$$

is eigenvector for eigenvalue Λ and matrix A.

The Matryoshka Property and Simplification

Matryoshka Property

Hypotheses:

$A_{i,.}$ is an increasing function (of the column index)

$B_{.,j}$ is an increasing function (of the line index)

Which can be written as

(L) $A_{i,k} \subset A_{i,k+1}$ for k=1 to M-1

and

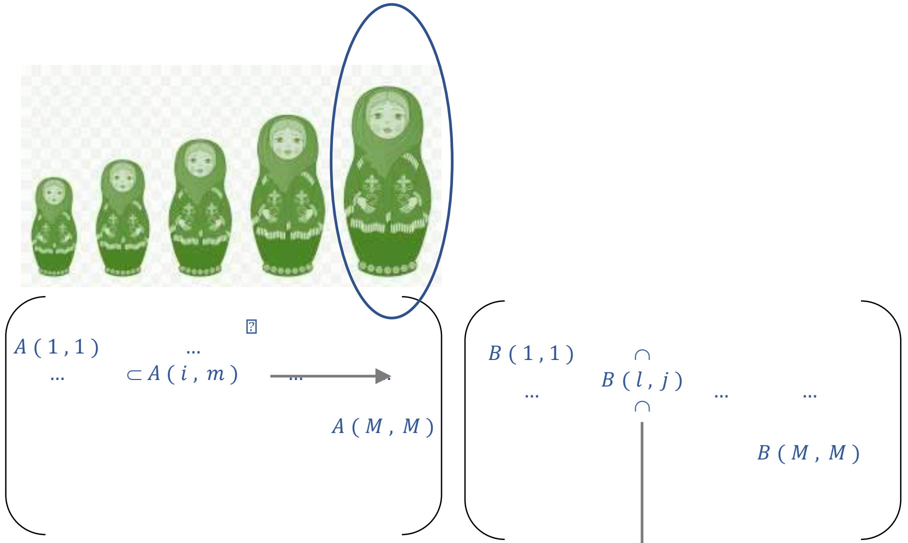
(C) $B_{k,j} \subset B_{k+1,j}$ for k=1 to M-1

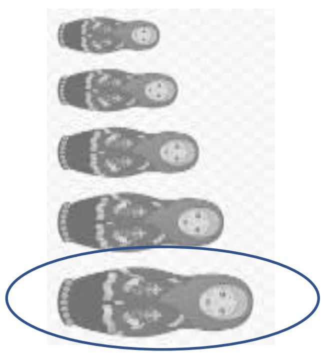
Hence

$A_{i,k} \times B_{k,j} \subset A_{i,k+1} \times B_{k+1,j}$ for k=1 to M-1

and

$$\bigcup_{k} A_{ik} \times Bkj = A_{iM} \times B_{Mj}$$



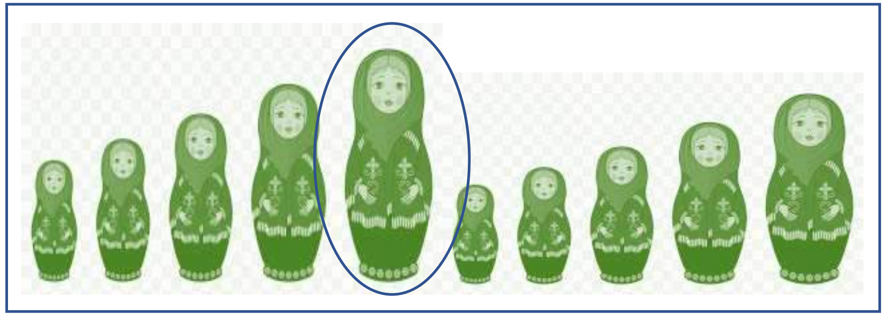


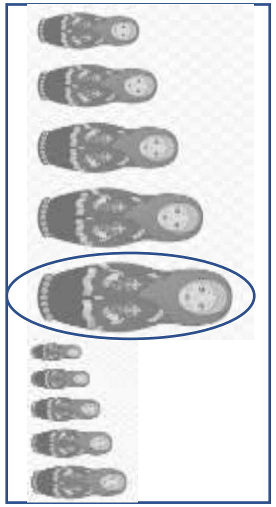
Generalised Matryoshka

Product Simplification Theorem

Under the combined condition
($L' \wedge C'$) $\exists$ m' such that $A_{i,k} \subset A_{i,m'}$ and $B_{k,j} \subset B_{m',j}$ for k=1 to M
then the product term $C_{i,j}$ *simplifies:*

$$C_{ij} = \bigcup_{k} Aik \, \mathbf{X} \, Bkj = A_{im'} \mathbf{X} \, B_{m'j}$$





Polynomials

A polynomial can be applied in a straightforward manner to a classical matrix M

$$P(X) = \sum_{k=0}^{D} a_k X^k$$

$$P(M) = \sum_{k=0}^{D} a_k M^k$$

Let us now consider a matrix of sets M.

For sets, "multiply by a_k" translates into **"set-matrix-multiply by $\mathbb{1}(A_k)$"**

We want to perform a set reunion. Let us look at physical dimensionality. Assume that the terms of M are sets in E, the highest degree term is M^D with set-matrix-coefficient $\mathbb{1}(A_D)$.

Let us assume that $A_D \in \mathrm{E}$

If we require $A_k \in E^{D+1-k}$, every term of matrix of sets $\mathbb{1}(A_k)$. M^k is in E^{D+1}

Then we can perform the reunion of all these terms in the same set E^{D+1}

$$P(M) = \bigcup_{k=0}^{D} \mathbb{1}(A_k)\ X\ M^k$$

Note that in the case of an eigenvector V for eigenvalue Λ

$$P(M)\ \mathrm{X}\ V = \bigcup_{k=0}^{D} \mathbb{1}\left(A_k \Lambda^k\right) \mathrm{X}\ V$$

Anonymisation of data carried by a Matrix of sets

Define matrix K(Λ_1,… Λ_M) the matrix formed of columns of Λ_k

Call it K for convenience

For any matrix of sets D containing data personal/private to person i and person j at position (i,j), we can anonymise the sets D(i,j) by performing

D’ = K^t x D x K

Let us start with C = D x K

$$C_{i,j} = \bigcup_k D_{ik} \times \Lambda_j$$

$$D'_{i,j} = \bigcup_k \Lambda_i \times C_{kj}$$

$$D'_{i,j} = \Lambda_i \times \bigcup_{k,m} D_{km} \times \Lambda_j$$

In this Matrix of Sets

$$\bigcup_{i,j} D_{ij}$$

Data cannot be immediately traced to specific people i and j they relate to
Hence Anonymisation has been performed by transforming D’ into D, with D’ = K^t x D x K

Scanning Use Cases for Matrices of Sets

Method

Harvest the fields of Matrix Crops

- Look for matrix based models
- Assess how Matrices of Sets could be used instead

Routing using Graphs

-Containers transport and logistics models with Matrices of Sets: -enabling digital efficiency gains for freight transport & logistics

Conference: International Association of Maritime Economists (IAME) 2021, Rotterdam

Renaud Di Francesco, Jaouad Boukachour, Amina el Yaagoubi, Mohamed Charhbili

Economics Models

Maritime Economics Computable Models using Matrices of Sets: study cases of 1) economics of routing for multimodal transport, 2) expression of preference across heterogeneous dynamic baskets Applications illustrating the efficiency of economic modelling with Matrices of Sets

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Renaud Di Francesco ,Pia Di Francesco-Isart

Collaborative Filtering

For online business and social network knowledge extraction

Two Matrices:

(i) **Reaction** of user i to product/service stimulus j

(ii) **Similarities** across products/stimuli (product i x product j) incorporating in-depth knowledge, semantics, features

Algorithmic principles

For user i identify users i in (i) with a number of similar preferences (similarity of products is in (ii)) and from these observe all products j not yet known to i or observed on i (M(i,j)=∅), then among those select those which either:*

*-**are farthest away** from existing preferences known of user i*: a "surprise" emphasis (although balanced with "acceptability likelihood" obtained through (ii))*

*-**are closest** to existing preferences of i*: a "comforting" choice*

Such selection is made using (ii).

Matrices of sets record very flexibly data on preferences expressed (comments/text, stars, actions, etc), as well as on the semantics of products/services/stimuli

Drug Target Identification: similar algorithms

Next Steps

Gram-Schmidt Matrices of sets

Correlations of vectors of sets

$$G_{i,j} = \langle \boldsymbol{X}_i, \boldsymbol{X}_j \rangle$$

$$G_{i,j} = \bigcup_k \boldsymbol{X}_{i,k} \times \boldsymbol{X}_{k,j}$$

To be studied

L-sets for extending Matrices of Sets with negative and positive sets

L-sets (reference to Loeb's sets with negative number of elements)

To interpret L-sets in simple terms, accounting paradigms of active and passive, or revenue and debt can be used. For each of us, there are

(i) Sets of what we have: -inventory of our possessions

(ii) Sets of what we do not have and need or want: -our shopping list

First explorations have illustrated how "+sets" and "-sets" can compensate each other, to remedy the unbounded growth of Matrices of Sets through multiplication.

References

i. Presentation Matrices of Sets: Applications, Reductions and Extensions to L-sets

ii. Presentation Matrices of Sets, complete tutorial with use cases

iii. Sets with a negative number of elements, Daniel Loeb, *Advances in Mathematics*, 91 (1992), 64–74.

iv. Containers transport and logistics models with Matrices of Sets: -enabling digital efficiency gains for freight transport & logistics

Conference: International Association of Maritime Economists (IAME) 2021, Rotterdam

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v. Maritime Economics Computable Models using Matrices of Sets: study cases of 1) economics of routing for multimodal transport, 2) expression of preference across heterogeneous dynamic baskets Applications illustrating the efficiency of economic modelling with Matrices of Sets

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Supplementary Material

Applications

Logistics
Economics
Generation/Evolution

Extensions

L-sets:

“+sets” {a,b,…} of what you have
“-sets” {bread, butter, veg…} of what you do not have and need

Container Logistics

Moving Containers Models, e.g. [Sönke Hartmann 2002, for Hamburg Harbour]

- Transport modes are either
- Transport mode >> transport mode size
- Time management

Intelligent Container [Mohamed Yassine Samiri 2018, for Le Havre harbour]

- Adaptive PRIoritizing Container Inspection (APRICOIN)

2017

2002

2018

Optimal Placement and Retrieval of Containers e.g. [Yachba Khadidja 2017]

- This is operational research
- Aiming at categorising containers adequately (Ex: safety for hazardous material, likely retrieval time before departure, storage duration, etc)

Container Logistics:
-Developing programmes with Matrices of Sets

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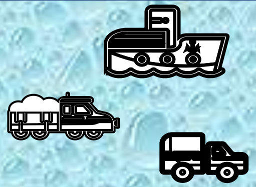
Programming with Matrices of Sets?

Optimal container placement algorithms with Matrices of Sets?

Geographic Information Analysis & Visualisation with Matrices of Sets?



Candidate use of Matrices of Sets in Economics

Basket allocations with shortage, management of preferred goods

Our IAME2021 Economics article describes an algorithm to manage shortage in supply using this model

Economics:
Use preference of baskets instead of utility

Auction-like model used in Internet marketplaces:

Agent i (buyer) is asked to express preferences, in the form of **baskets of goods (or services)** $\mathbf{B_{i,j}}$ with M choices in j, **in decreasing order of preference**, from the first preferred basket to the least preferred one:

$B_{i,1} > B_{i,2} > \ldots > B_{i,M}$

Then a Matrix of set is formed:

$$\boldsymbol{B} = \begin{pmatrix} B_{1,1} & \ldots & B_{1,M} \\ \ldots & B_{i,j} & \ldots \\ B_{N,1} & \ldots & B_{N,M} \end{pmatrix} \quad \leftarrow \text{line } i \text{ (agent, a buyer)}$$

↑

column *j* (preference rank)

NxM Matrix of Sets describing "agent" X "order of preference"

Collaborative Filtering with Matrices of Sets

-Encapsulating observed reaction of Customers to Products

Economics:

Use complete information on customer's views on products instead of scores

Collaborative Filtering (CF) is used in its classical form (classical matrices) for product recommendations on major Internet marketplaces (Ricci F. et al, 2015), along the somewhat rough idea that you may like or consider products liked or bought by "users behaving in a way similar to you", with whatever assumptions this may entail. Let us show how such a model of economic relevance since it is implemented for commercial benefit, can be generalised using information encapsulated in Matrices of Sets.

In this use case, **agents are indexed by i**, **goods/products by j**, **B(i,j) aggregates multiple types of information on the reaction of agent i to product j** if any (otherwise empty set): purchased (or not), as comments/views expressed and recommendations (text), other details:

$$\boldsymbol{B} = \begin{pmatrix} B_{1,1} & \dots & B_{1,M} \\ \dots & B_{i,j} & \dots \\ B_{N,1} & \dots & B_{N,M} \end{pmatrix} \quad \leftarrow \text{line } i \text{ (agent)}$$

$\uparrow$

column j (product)

NxM Matrix of Sets as "agent" X "product" reaction for Collaborative Filtering

Collaborative filtering with Matrices of Sets

-Correlations between Products

Economics:
Use complete information on product differences/commonalities instead of similitude scores

A second matrix d studies how close products m and n are from one another, with a matrix term d(m,n) for a certain distance or similarity measure. We generalise this, with Matrix of Set C (for Content), which has terms

$$C_{m,n} = C_{+\ m,n}\ \cup\ C_{-\ m,n}$$

where + aggregates positive views/commonalities for (m,n) (scores, stars, text, etc) and C- negative views/divergent aspects (scores, thumbs down, text, etc).

$$\boldsymbol{C} = \begin{pmatrix} C_{1,1} & \dots & \boldsymbol{C}_{1,M} \\ \dots & C_{i,j} & \dots \\ C_{N,1} & \dots & C_{N,M} \end{pmatrix} \quad \leftarrow \text{line } i \text{ (product)}$$

$\uparrow$

column j (product)

MxM Matrix of Sets as "product" similarity for Collaborative Filtering

This matrix encapsulates in a flexible manner commonalities and differences between products. Optimal suggestions of Products to Customers can be built using this model comprising two matrices of sets

Candidate use of Matrices of Sets in Multi-generational Models

Generational Models

$$\boldsymbol{G} = \begin{pmatrix} G_{1,1} & \dots & G_{1,M} \\ \dots & G_{i,j} & \dots \\ G_{N,1} & \dots & G_{N,M} \end{pmatrix} \quad \leftarrow \text{line } i \text{ (parent)}$$

$\uparrow$

column j (child)

Let us build a Generational Matrix of Set G, with “parent” as line i=1 … N and “child” as column j=1 …M

Precisely:

i is parent of j (and j is child of i) if and only if $G_{i,j}$ is non-empty, and in that case, the **set $G_{i,j}$ contains information on “parent” i and “child” j relevant to the problem addres**sed.
G is a Genealogical Generalised Adjacency Matrix of Sets (GGAMS ☺).

Generational Models

-"Evolutions"?

Define

$$\Gamma^{(t)} := \prod_{\tau=t_0}^{t} G^{(\tau)}$$

This matrix of sets encapsulates the genealogical line from parents "in t_0 generation" to children "in t generation". It aggregates the information set linking "parent" and "child".

Note that multi-parental relations are possible ("child" can have up to n "parents") under this format of description, which is suitable for any family definition and broader use (e.g. "parent process" vs "child process").

To give pedestrian examples, equipped with such generational matrices of sets, one could answer queries such as "who is the first not to have a medical profession, among the descendants of a medical professional?", and analyse the factors driving to such a fact, or "who is the first university graduate among the descendants of a farmer?" with associated factors around this event.

Matrix of sets: size control

Matrices of sets, when multiplied, perform unions of terms, with a likely growing size after each multiplication (with the exception of void terms leading to a void set contributed the union).

It is therefore important to consider mechanisms to reduce their size. A candidate method is Quantization as in Information Coding.

In general terms let us observe that we can define the inclusion between matrix of set A and matrix of set B by the property that

-for each line i and column j, $A_{i,j} \subset B_{i,j}$,

then we will write

$A \subset B$

A down-sampling of B into A can be defined as any set A strictly included in B. One can define a homogeneous down-sampling d over all positions at line i and column j if for each i, j,

A is such that Card($A_{i,j}$) is the quotient of the Euclidean division of Card($B_{i,j}$) by d, with the associated remainder denoted by $r_{i,j}$:

$Card(B_{i,j}) = d.Card(A_{i,j}) + r_{i,j}$

with $A_{i,j} \subset B_{i,j}$

Matrix of sets: size control

Inhomogeneous down-sampling schemes of B into A with ratio d, can be defined as well, with more relaxed constraints such as

$$\sum_{i,j} Card\, B(i,j) = \mathrm{r} + \mathrm{d} \sum_{i,j} Card\, A(i,j)$$

thus forming a Euclidean division by d with remainder r, on the sums of Cardinals, while keeping for each i, j

A(i,j) $\subset$ B(i,j)

Learning and forgetting

This section refers to signal processing concepts. Matrices of sets A are observed. The first time t a matrix of set is observed, it is denoted by A(t | t) and is stored.

Over time, say at t+n, one may want to replace A(t | t) by a subset A(t+n | t) of smaller size, a summary of A(t | t).

Iterating k steps, at t+kn, the summary becomes A(t+kn | t).

The information stored from time origin t_0 to present is then summarised periodically every n, at time t_0 +kn

L-sets: as in (real) life,and recognising D. Loeb's pioneering work)

L-set A decomposed into

-what you have A+

-what you have not A-

The pioneering article by D. Loeb of 1992 associates to elements of a set a number in $\mathbb{Z}$, which is positive for classical sets and negative for what is called new sets in the article. Let us denote by letter q this quantity, also by reference to the classical electrical charge notation which can be positive or negative. For a set S of elements a, and a function q from S to $\mathbb{Z}$, we define another set A as:

$A = \{ (a,q(a))/\forall a \in S, \text{ such that } q(a) \neq 0\}$

Note that A is the graph of the function q on S. Let us adapt notations to make them easier to use, and denote the quantity/charge function by q_A and the reference set of A by S_A

A can be decomposed into A+ and A- as follows:

$A+ = A \cap S \times \mathbb{N}$

$A- = A \cap S \times (-\mathbb{N})$

Those two sets have an empty intersection since elements with quantity/load zero are not in A.

L-sets: as in (real) life,and recognising D. Loeb's pioneering work)

L-set A decomposed into

-what you have A+

-what you have not A-

For two L-sets A and B, let us now define the reduced product of A+ by B+:

-the Cartesian product of A+ and B+ is

$C = \{((a,q_A(a)), (b,q_B(b)))/ \ \forall a \in S_A, \text{ such that } q_A(a) \neq 0; \ \forall b \in S_B, \text{ such that } q_B(b) \neq 0\}$

Let us define the L-set product of A+ and B+, and A-, B-, as

$L(A+,B+) = \{((a,b), q_{A+}(a)q_{B+}(b))/ \ \forall a \in S_{A+}, \text{ such that } q_{A+}(a) > 0; \ \forall b \in S_{B+}, \text{ such that } q_{B+}(b) > 0\}$

$L(A-,B-) = \{((a,b), -|q_{A-}(a)q_{B-}(b)|)/ \ \forall a \in S_{A-}, \text{ such that } q_{A-}(a) < 0; \ \forall b \in S_{B-}, \text{ such that } q_{B-}(b) < 0\}$

In this way we preserve results of products L(A+,B+) and L(A-,B-) staying in the same L-set category. We continue by defining L(A,B) as

L(A,B) = L(A+,B+) U L(A-,B-)

One could also use a formulation with complex numbers by denoting

$q_A = q_{A+} + i\, q_{A-}$

Then the real part of the product of the quantities/loads becoming

$Re(q_A(a)q_B(b)) = q_{A+}(a)\, q_{B+}(b) - q_{A-}(a)q_{B-}(b)$

for (a,b) in (A+)x(B+) U (A-)x(B-)

Let us leave the interpretation of the imaginary part (cross products +, -) for later.

L-sets: as in (real) life,and recognising D. Loeb's pioneering work)

L-set A decomposed into

-what you have, as A+

-what you have not, as A-

Possibility of compensation from +sets and –sets:

$$M''_{ij} = \bigcup_k L(M_{ik}, M'_{kj})$$

can be $\varnothing$

L(A+,B+) = {((a,b), $q_{A+}(a)q_{B+}(b)$)/ $\forall a \in S_{A+}$, such that $q_{A+}(a) > 0$; $\forall b \in S_{B+}$, such that $q_{B+}(b) > 0$}

L(A-,B-) = {((a,b), -|$q_{A-}(a)q_{B-}(b)$|)/ $\forall a \in S_{A-}$, such that $q_{A-}(a) < 0$; $\forall b \in S_{B-}$, such that $q_{B-}(b) < 0$}

L(A,B) = L(A+,B+) U L(A-,B-)

Let us now come to an adapted definition of the matrix product M" of matrices of L-sets M and M':

$$M''_{ij} = \bigcup_k L(M_{ik}, M'_{kj})$$

Let us consider the case of 2x2 L-matrices, with an illustrative example how compensation of "+sets" and "–sets" play out:

M"= $\begin{pmatrix} \{(a,1),(b,-1)\} & \{(c,1),(d,-1)\} \\ \{(e,1),(f,-1)\} & \{(g,1),(h,-1)\} \end{pmatrix}$ x $\begin{pmatrix} \{(a',1),(b',-1)\} & \{(c',1),(d',-1)\} \\ \{(e',1),(f',-1)\} & \{(g',1),(h',-1)\} \end{pmatrix}$

M"= $\begin{pmatrix} \{((a,a'),1),((c,e'),1)\} U \{((b,b'),-1),((d,f'),-1)\} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

For $M''_{11} = \varnothing$

it suffices to take

$(a,a') = (d,f')$ and (c,e')= (b,b')